On the Nature of Isotherms at First Order Phase Transitions
Lecture Notes for the Kac Seminar 2004

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1. Introduction

These lectures are devoted to the exposition of recent results about the nature of the singularity of the pressure and the free energy at a first order phase transition [FrPf1], [Fr] and [FrPf2]. The main results of these works are summarized in [FrPf3]. There are two principal results, one concerning the nature of the singularity of the pressure at a first order phase transition, which is proved at low temperature for lattice models with finite state space, finite range interactions, with two periodic ground states verifying the Peierls condition. The second one constitutes the bulk of the PhD thesis of Sacha Friedli; it concerns the van der Waals limit of Ising models, and how analyticity of the free energy is restored in this limit at a first order phase transition point. This section is devoted to a selective historical introduction to the subject.

1.1. **1869-1875.** Andrews' Bakerian Lecture to the Royal Society in 1869 was entitled "On the Continuity of the Gaseous and Liquid States of Matter" [An]. This paper is famous for the first experimental proof of the existence of the critical temperature, a term coined by Andrews himself in this paper. For the first time precise measurements of several isotherms for carbon dioxide were performed above, below and at the critical temperature. Andrews deduced that *the ordinary gaseous and ordinary liquid states are, in short, only widely separated forms of the same condition of matter, and can be made to pass into one another by a series of gradations so gently that the passage shall nowhere present any interruption or breach of continuity.* In 1822 Cagniard de la Tour had already found that if ether, alcohol or water were heated in a sealed tube the volumes of the liquids increased by about two- to four-fold, but eventually the liquid was apparently converted into gas[1]. But he had no clear idea of the significance of this result. In my lectures it is not the critical temperature T_c which is my interest, but the isotherms for temperatures T (well) below T_c, where there are sharp breaks at the gas and liquid ends of isotherms when the phenomenon of condensation takes place.

In 1871 James Thomson wrote a speculative paper [Th] about the isotherms of a simple fluid[2]. After summarizing the experimental results of Andrews [An], proving the existence of the critical point and the fact that one can pass from the gaseous state to the liquid state by *a course of continuous physical changes presenting nowhere any interruption or breach of continuity*, he wrote *it will be my chief object in the present paper to state and support a view which has occurred to me, according to which it appears probable that, although there is a practical breach of continuity in crossing the line of boiling-points from liquid to gas or from gas to liquid, there may exist, in the nature of things, a theoretical continuity across this breach having some real and true significance. This theoretical continuity, from the ordinary liquid state to the ordinary gaseous state, must be supposed to be such as to have its various courses passing through conditions of pressure, temperature, and volume in unstable equilibrium for any fluid matter theoretically conceived as*

[1]See beginning of [An].

[2]In 1871 Maxwell was writing his book Theory of Heat and he gave an account of the works of Andrews and Thomson. Thomson's ideas are discussed at p.124-127 in [M1], in the chapter which is devoted to the isothermal lines, as well as to the experiments of Cagnard de la Tour and of Andrews. There are few concrete arguments in this paper, and its importance cannot be compared to van der Waals' dissertation [vdW1].

homogeneously distributed while passing through the intermediate conditions. Such courses of transition, passing trough unstable conditions, must be regarded as being impossible to be brought about throughout entire masses of fluids dealt with in any physical operations. Whether in an extremely thin lamina of gradual transition from a liquid to its own gas, in which it is to be noticed the substance would not be homogeneously distributed, conditions may exist in a stable state having some kind of correspondence with the unstable conditions here theoretically conceived, will be a question suggested at the close of this paper ....

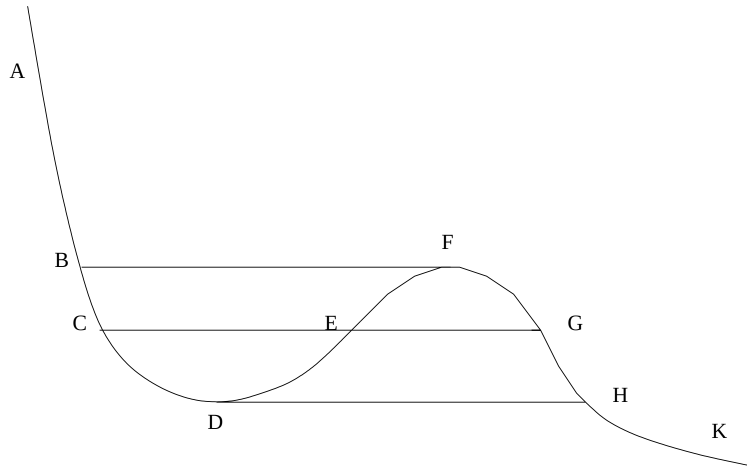

Figure 1: Thomson's isotherm below the critical temperature, pressure as function of volume.

Below the critical temperature, the theoretical isotherms proposed by Thomson have a minimum and a maximum. Their interpretation, in 1871, is the one which one finds in many text-books, and which is still taught today[3]. There is a well-defined pressure, say corresponding to states C, E and G, where the fluid can be in two different stable (equilibrium) states C and G, and such that one can pass from one state to the other in a reversible way. This corresponds to a first order phase transition point. What is still missing is the determination of that pressure[4]. Maxwell published his "equal area rule" only in 1875. All states along the isotherm between B and C correspond to stable liquid states, and similarly all states between

[3]See for example [Ca] chapter 9, or [CoM] chapter 8.

[4]In a correspondence with Thomson, Maxwell wrote (13 July 1871 [Ha2] 668-669): *I should like to hear from you if the proof I send gives a fair account of what Andrews and you have done and more particularly if you have told me anything in confidence that you have not yet published mark it out. I hope however that you will publish some of what you told me for the speculation seemed of the fertile kind.* [...]. *The next difficulty is What determines the true boiling temperature of the steam which is found to be so constant?* In his reply (21 July 1871 [Ha2] 670-671) Thomson wrote: *I think it is* not *possible for the substance at the pressure indicated by B to pass into the gaseous state & that if the liquid is in contact with its vapour at this pressure it is* really found that the liquid will not *begin to pass into the gaseous state. On the contrary I think under the circumstances stated it will all go down to the liquid state.* Then Thomson made a similar statement for the state H. Concerning the question of Maxwell about the true boiling temperature, Thomson wrote *I will answer rather a corresponding question* [...]*:* What determines the true boiling pressures of steam which is found to be so constant for any given temperature? *I reply:- There is just one intermediate point of pressure between the pressure at F and the pressure at D, at which the liquid and its gas can be present together in contact with one another.*

H and G are stable vapour states. The states between C and D exist as homogeneous liquid states, but are not equilibrium states. They represent superheated liquid states and they are metastable states. Such states were experimentally observed in 1871. Similarly the states between G and F represent supercooled vapour states and are metastable. Later, in his treatise on Thermodynamics §27 [Pl] Planck gives the interpretation of the isotherms in the following terms: *At all times, it is possible to follow the isotherm beyond the point G towards the point F, and to prepare a so-called supersaturated vapour. Then only a more or less unstable condition of equilibrium is obtained, as may be seen from the fact that the smallest disturbance of the equilibrium is sufficient to cause an immediate condensation. The substance passes by a jump into the stable condition. Nevertheless, by the study of supersaturated vapours, the theoretical part of the curve also receives a direct meaning.* On the other hand, Maxwell did not attribute any physical meaning to the unstable part of the isotherm between D and F (see [M1] p.125), contrary to Thomson. In this respect van der Waals wrote in [vdW1][5]: *The idea of joining C and G by a straight line, as is done by Maxwell, is not a happy one.* Since for these states the pressure is increasing as the volume increases, they cannot be realized as stable (homogeneous) states, but Thomson thought that they might be realizable at the interface between gas and liquid. I shall come back to this important point when concluding this subsection.

Van der Waals published his famous dissertation in 1873 [vdW1], whose title in English is almost identical to the title of Andrews' Bakerian Lecture in 1869: "On the Continuity of the Gaseous and Liquid States." This fundamental work, its consequences and later developments are analyzed thoroughly by Rowlinson in his book [R2], where an English translation of van der Waals' dissertation is also given. See also [Kl]. It is in this work that appears the famous equation of state, which can be written

$$\Big(p + \frac{a}{v^2}\Big)\big(v - b\big) = kT \tag{1.1}$$

In that formula p is the externally applied pressure, v is the specific volume, or v^{-1} the density, a/v^2 $(a > 0)$ is the molecular pressure arising from attraction between the molecules. The first factor is interpreted as the total effective pressure. The factor b is four times the effective volume of the molecule, so that the second factor is the effective volume, per particle, within which the molecules can move. The right hand side is proportional to the kinetic energy per particle, and T is the absolute temperature. Van der Waals explained Andrews' results on the continuity of the gaseous and liquid states by his famous equation, and gave a solid theoretical foundation to Thomson's speculations. There exists a critical temperature T_c such that for $T > T_c$ there is only one real solution for v, given p and T. On the other hand, if $T < T_c$ there are three real solutions, and qualitatively the isotherms are similar to those of Thomson.

Equation (1.1) was based on Clausius' virial theorem, which relates the kinetic energy of molecules to forces acting on them [Cl]. If $\mathbf{v}_i$ is the velocity of particle i at position $\mathbf{x}_i$, and $\mathbf{F}_i$ the resultant of all forces acting on particle i, then the average over long times of the total kinetic energy (*vis viva*) is equal to the average of the virial,

$$\frac{1}{2}\langle\sum_i m_i v_i^2\rangle = -\frac{1}{2}\langle\sum_i \mathbf{x}_i \cdot \mathbf{F}_i\rangle\,.$$

[5]English translation, [R2] p.196.

If the internal forces are central forces, then this expression becomes[6]

$$\frac{1}{2}\langle\sum_i m_i v_i^2\rangle = \frac{3}{2}pV + \frac{1}{2}\sum_{i,j} r_{ij}\phi(r_{ij})\,.$$

In this formula V is the volume of the vessel containing the particles, p the pressure, r_{ij} the distance between particles i and j, and $\phi(r_{ij})$ the intensity of the force between particles i and j, which is positive when the force is attractive.

This work received immediate recognition and Maxwell wrote a long review in Nature in 1874 [M2]. At the beginning of his review Maxwell wrote: *That the same substance at the same temperature and pressure can exist in two very different states, as a liquid and as a gas, is a fact of the highest scientific importance.* A large portion of the second part of Boltzmann's Lectures on Gas Theory is devoted to van der Waals' theory [Bol]. Planck wrote in his treatise on Thermodynamics [Pl], in paragraph §24: *To van der Waals is due the first analytical formula for the general characteristic equation, applicable also to the liquid state. He also explained physically, on the basis of the kinetic theory of gases, the deviations from the behaviour of perfect gases.* Later, in paragraphs §26, §27 and §28, Planck discusses in details a modification of equation (1.1) due to Clausius, which fits observations on the compressibility of gaseous and liquid carbon dioxide at different temperatures fairly well. For a recent account of van der Waals' equation see [ELi], which also contains the derivation of (1.1) due to Ornstein from statistical mechanics [Or] (Leiden dissertation). It is based on the idea that the interaction pair potential between particles consists in a repulsive hard-core that is short range and an attractive, weak, long-range part. The theory is what is called today a mean-field type theory.

In the same year 1873 Gibbs published his important paper *A Method of Geometrical Representation of the Thermodynamic Properties of Substances by Means of Surfaces* [G1], where he gave a geometric characterization of the phase diagram by introducing the energy-volume-entropy surface, which he called the thermodynamic surface of the body[7]. Specifically he discussed the surface

$$u = u(s, v) \quad u \text{ the energy, } s \text{ the entropy, } v \text{ the volume.}$$

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[6]See [Cl] or [LLi] pp. 113-114. The virial theorem is properly a theorem of analytic mechanics. The term *virial* is due to Clausius [Cl]. It come from Latin *vis* (force). In [Cl], Clausius considers the case of a large number of particles. The motion of the particles is stationary in the following sense: *By stationary motion I mean one in which the points do not continually remove further and further from their original position, and the velocities do not alter continuously in the same direction, but the points move within a limited space, and the velocities only fluctuate within certain limits.* Clausius made the following important remark concerning the time average: *it is not necessary to take the mean value of* $r\phi(r)$ $[r_{ij}\phi(r_{ij})]$ *for each pair of atoms, but the values of* $r\phi(r)$ *may be taken for the precise position of the atoms at a certain moment, as the sum formed therefrom does not importantly differ from their total value throughout the course of the individual motions. Consequently we have for the internal virial the expression*

$$\frac{1}{2}\sum_{i,j} r_{ij}\phi(r_{ij})\,.$$

The term $\frac{3}{2}pV$ is the result of the integration of the external forces over the boundary of the vessel, after using Gauss' divergence theorem ($\mathrm{div}(\mathbf{x}) = 3$).

[7]Gibbs made explicit reference to Thomson's paper [Th], who also introduced a surface, but for different quantities, namely the temperature, volume and pressure, so that the isotherms are level-lines of that surface.

The pressure and the temperature of the state, which is represented by a point of the surface, give the directions of the tangent plane at that point,

$$p = -\frac{\partial u}{\partial v} \quad \text{and} \quad T = \frac{\partial u}{\partial s}\,.$$

Gibbs remarked that this mode of representation applies also to the case in which the system is not in a homogeneous state, for example, a state representing a mixture of vapour and liquid *in equilibrium*. The surface obtained in this way is called *thermodynamic surface*. He divided this thermodynamic surface into two parts: the primitive surface, whose points correspond to homogeneous states, and the derived surface, whose points do not correspond to homogeneous states[8]. His main point is that there is a simple geometric relation between the primitive and derived surfaces, which is a consequence of the fact that the volume, entropy and energy of the whole body are equal to the sums of the volumes, entropies and energies respectively of the parts, while the pressure and temperature of the whole are the same as those of each of the parts. Knowing the primitive surface, one can reconstruct the derived surface by rolling a tangent plane on the primitive surface. An important point is that the form of the primitive surface is such that the rolling tangent plane does not cut it; the thermodynamic surface is convex. He showed that the thermodynamic equilibrium between gas and liquid is achieved at the pairs of points of contact with the primitive surface of a rolling tangent plane. In the context of these lectures, the following passage of Gibbs' paper is relevant[9]: *If there is no gap in the primitive surface, there must evidently be a region where the surface is concave toward the tangent plane in one of its principal curvature at least, and therefore represents states of unstable equilibrium ....* This hypothesis, *there is no gap in the primitive surface*, leads to similar results as those of Thomson. Maxwell was enthusiastic about Gibbs' surface. In a letter to T. Andrews[10] he wrote: *I think such graphical methods are better fitted for purely conjectural applications of the principle of continuity beyond the range of experiment than any empirical formulae.*

At the end of 1874 Maxwell formulated the "equal area rule". He announced his result to G. Tait in the following terms[11]: *In James Thomsons figure of the continuous isothermal show that the horizontal line representing mixed liquid and vapour cuts off* equal areas *above & below that curve. Do this by Carnots cycle. That I did not do it in my book shows my invincible stupidity.* This thermodynamic argument was published in 1875 in [M3]. The value p^* of the pressure, for which there is a plateau in the isotherm, is determined by the condition

$$p^*(v_g - v_l) = \int_{v_l}^{v_g} p(v)\, dv\,,$$

$p(v)$ being the equation of the isotherm given by equation (1.1). Since the pressure is given (up to the sign) by the derivative of the Helmholtz free energy, $f = u - Ts$, which gives the maximum work that can be extracted from the system along any

[8]For example, in the case of the coexistence of vapour and liquid, a point of the derived surface represents an inhomogeneous (macroscopic) state, where a portion α of the system is in the vapour phase and a portion $1-\alpha$ is in the liquid phase. The vapour and liquid states are represented by two different points of the primitive surface. There is *breach of continuity.*

[9][G1] p.45.

[10]15 July 1875, [Ha3] pp. 236-238.

[11]28 December 1874 [Ha3] 155-156; see also [Ha3] 157-158.

isotherm, Maxwell's rule is equivalent to take first the convex envelope f^* of the Helmholtz free energy f, and then to take the derivative, so that the pressure of the isotherm with the plateau is given by

$$-p = \frac{\partial f^*}{\partial v}\,.$$

To summarize these fundamental results obtained in this short period of time, one can perhaps say that a salient feature of these experimental and theoretical developments is the emphasis on the *idea of continuity*[12], which finds an experimental basis in the work of Andrews about the existence of the critical point, and a firm theoretical basis in the work of van der Waals. The van der Waals isotherms are analytic curves. For each fixed value of the temperature below T_c Maxwell's rule gives the (unique) value of the pressure for which vapour and liquid coexist as equilibrium phases; thus, the equilibrium isotherms at low temperature have three distinct analytic parts, the middle flat part defined through Maxwell's rule corresponds to physical situations where both the vapour and liquid coexist as equilibrium phases. There are analytic continuations for the two other parts, which are given by the van der Waals isotherm, and the parts of these analytic continuation where $\frac{\partial p}{\partial v} < 0$ are interpreted as superheated liquid states, respectively undercooled vapour states. Even the more problematic part of the analytic isotherm, between the minimum and the maximum of the isotherms, where $\frac{\partial p}{\partial v} > 0$, plays a role in the mechanical theory of surface tension developed by Fuchs and Rayleigh[13], and in the thermodynamic theory of capillarity of van der Waals [vdW2]. The theory of van der Waals has been revisited later by Cahn and Hilliard [CHi]. Suppose that the Helmholtz free energy is written as a function of the density ρ, at fixed (low) temperature, $\psi = \psi(\rho)$. Here ψ is the analytic free energy, which is non-convex in the two-phase region. Thomson's idea, that the non convex part of $\Psi(\rho)$ might be physically realizable in the interface between equilibrium phases, is implementing as follows (see e.g. [W1]). The surface tension, which is the excess of free energy in the inhomogeneous system,

[12]Maxwell, in his letter to Andrews quoted above, refers to the *principle of continuity*. Duhem, who did not accept the atomistic model of thermodynamics, in contrast to Boltzmann, Clausius, Maxwell, van der Waals and others, considers in his epistemologic treatise La théorie physique, son objet - sa structure ([Du]), that *la théorie de la continuité de l'état liquide et de l'état gazeux*, is an important theory of Physics ([Du] p.138-139). The van der Waals isotherms are analytic. Absence of gap in Gibbs' primitive surface is related to Thomson's ideas. Furthermore, Andrews' paper [An] ends as follows: *We have seen that the gaseous and liquid states are only distant stages of the same condition of matter, and are capable of passing into one another by a process of continuous change. A problem of far greater difficulty yet remains to be solved, the possible continuity of the liquid and solid states of matter.* [...] *for the present I will not venture to go beyond the conclusion I have already drawn from direct experiment, that gaseous and liquid forms of matter may be transformed into one another by a series of continuous and unbroken changes.* The following quotation of Herschel's *Preliminary Discourse on the Study of Natural Philosophy* is also worth mentioning in relation with this idea of continuity (see [R2] p.4). *Indeed, there can be little doubt that the solid, liquid and aëriform states of bodies are merely stages in a progress of gradual transition from one extreme to the other, and that, however strongly marked the distinctions between them appear, they will ultimately turn out to be separated by no sudden or violent line of demarcation, but shade into each other by insensible gradations. The late experiments of baron Cagnard de la Tour may be regarded as a first step towards a full demonstration of this (§199).* The reference to §199 of his book is to "that general law which seems to pervade all nature - the law, as it is termed, of continuity, and which is expressed in the well-known sentence "Natura non agit per saltum".

[13][R2] p.6.

is assumed to be a functional of the density profile[14] $\rho(x,y,z)=\rho(x)$, $x\in\mathbb{R}$, which can be written as

$$\int \Psi(\rho(x))\,dx := \int \Big[\psi(\rho(x)) + \frac{1}{2}A\rho'(x)^2\Big]\,dx\,.$$

This functional is the sum of two terms; $\psi(\rho(x))$ describes a completely homogeneous state (perhaps unstable) at the local density $\rho(x)$, and $\frac{1}{2}A\rho'(x)^2$, which is a term due to van der Waals [vdW2], is a first correction for deviations from uniformity. The density profile between the vapour and the liquid phases, which are at equilibrium, is assumed to minimize the surface tension subject to the boundary conditions imposed to the system. Therefore, the non convex part of the free energy corresponding to the metastable and unstable states plays a significant role in the above functional. Without the second term, there would be no non-trivial density profile, since the minimum of the functional would be attained by a profile with $\rho(x)=\rho_g$ if $x<x^*$, and $\rho(x)=\rho_l$ if $x>x^*$; the point x^*, which gives the position of the interface, is determined by the boundary conditions and the quantity of vapour, respectively liquid, in the system.

Many works nowadays, about phase transitions and interfacial phenomena, are based on functionals of the above type, with a non convex part $\psi(\rho)$, which is assumed to be an analytic continuation of the equilibrium free energy (see e.g. [La2].) Maxwell's equal area rule or Gibbs's convex envelope of the free energy are used to determine the phase coexistence points. Our main result is that, *for short-range interaction potentials*, at least for a large class of models, *such functionals cannot be derived from first principles of Statistical Mechanics: there is no analytic continuation of the free energy (and equilibrium isotherms) at a first order phase transition.*

1.2. **1937-1952.** I shall not discuss the period from 1875 to 1937, although many important papers in relation with phase transitions appeared during that period. Systematic corrections for the law of perfect gases were studied from the virial expansion. Ferromagnetism was extensively studied and in particular mean-field type theories were developed (Curie-Weiss model, Bragg-William approximation). Lattice models were used to study phase transitions. For this period, see the essay of Rowlinson [R2]. One of the achievements of nineteenth century physics was the development of the statistical (i.e. microscopic) basis of thermodynamics, which owes its origin to the desire to explain the laws of thermodynamics from mechanical principles, and of which Clausius, Maxwell and Boltzmann are to be regarded as the principal founders. But it is Gibbs's work, in particular his monograph, *Elementary Principles in Statistical Mechanics*, [G2], published in 1901, which is the basis of our present formulation of equilibrium statistical mechanics. However, when this book appeared, statistical mechanics was facing one of its biggest problems in relation with the behaviour of various specific heats[15]. It is therefore remarkable that this

[14]One assumes that $\rho(x)\to\rho_g$ when $x\to-\infty$, and $\rho(x)\to\rho_l$, when $x\to\infty$. The system is translation invariant in the other two directions.

[15]Gibbs wrote (p.vii-viii of [G2]): *In the present state of science, it seems hardly possible to frame a dynamic theory of molecular action which shall embrace the phenomena of thermodynamics, of radiation, and of the electrical manifestations which accompany the union of atoms. Yet any theory is obviously inadequate which does not take into account of all these phenomena. Even if we confine our attention to the phenomena distinctly thermodynamic, we do not escape difficulties in as simple*

book gives the foundations of equilibrium statistical mechanics as we know it today. Our understanding of phase transitions since the beginning of the 20th century *is based on the very successful application* of the principles exposed in this monograph to a wide variety of physical problems. However this confidence on the principles developed in [G2] did not arise in one day, and the success of this fundamental approach was slow[16]. The fact that we can describe with the help of a *single mathematical expression, the partition function*, both the liquid and the gaseous phases, is a major step in our understanding of phase transitions.

The picture which emerges from the paper of Mayer, [Ma], to the paper of Yang and Lee, [YLe], is very different from the preceding one: the emphasis is on the *idea of singularity*. In a series of papers Mayer and his collaborators studied the phenomenon of condensation. The first paper of the series published in 1937, [Ma], prompted immediately several important papers, by Born and Fuchs [BF], Kahn's dissertation (1938) at Utrecht [Ka], Kahn and Uhlenbeck [KaU]. See also De Boer [dB1]. The paper of Mayer was discussed at the Van der Waals Centenary Congress in Amsterdam on November 1937. The results were presented by Born [B]. Born wrote: *I consider this work as a most important contribution to the development of van der Waals theory, which ought to be reported at this meeting, in spite of the fact that Mayer's methods are rather difficult to understand and his results not completely satisfactory.* About this report we can read in [BF]: *[it] was followed by a vigorous discussion on the question as to whether Mayer's explanation of the phenomena of condensation is correct. Doubts about this point were raised by the referee, because it is difficult to comprehend how a method of approximation such as that of Mayer, starting from the gaseous state, can lead to the discontinuity of the density on an isothermal curve which corresponds to condensation. The usual methods for treating the equilibrium of the two phases introduce the equation of state of both phases and derive the condition for their co-existence. Mayer's theory does nothing of this kind, but treats all possible molecular arrangements with their proper weight, as if there were only one phase. How can the gas molecules "know" when they have to coagulate to form a liquid or solids? Mayer's mathematical method is too involved to make this point quite clear*[17].

Indeed, a major problem, not touched by the theory of van der Waals, which deals with *pure homogeneous* states only, is the mechanism leading to condensation. Moreover, the horizontal part of the equilibrium isotherm corresponds precisely to a region where the system is inhomogeneous. Instead of describing the state of a system by its density only, Mayer tried to evaluate the (canonical) partition function of a system composed of N particles inside a volume V, starting from the expansion

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a matter as the number of degrees of freedom of a diatomic gas. It is well known that while theory would assign to the gas six degrees of freedom per molecule, in our experiments on specific heat we cannot account for more than five. Certainly, one is building on an insecure foundation, who rests his work on hypotheses concerning the constitution of matter.

[16] In his 1960 lectures [UF], Uhlenbeck (p.32-34) listed three basic questions, which one would like to answer on the basis of statistical mechanics: (a) the deviations from the ideal gas law, (b) the condensation phenomenon, and (c) the existence of a *critical temperature* T_{crit}. Then he wrote: *There are other general phenomena. At still smaller volume and probably at* any *temperature the substance solidifies, and has the corresponding solid-liquid and solid-vapour equilibria. But the explanation of these phenomena from the basic integral for* $Z(V,T,N)$ *[partition function] is still far from being accomplished* .... This is still true today!

[17] See also footnote 26.

of the partition function in terms of the cluster integrals (integrals of the Ursell functions over the phase space). He assumed that the cluster integrals are independent of the volume and are positive[18] below the critical temperature. In his computation he took into consideration *all* clusters; it is the statistics of the *large* clusters which are decisive for the phenomenon of condensation. This was an important and novel feature with respect to mean field type theories. One of the defects of the theory, however, was the impossibility of computing the isotherm of the liquid phase[19].

One can say that the essence of Mayer's theory consists in the study of the expansion of the pressure,

$$\frac{p(z)}{kT} = \sum_{l \geq 1} b_l z^l \quad (b_l \text{ are the cluster integrals})\,,$$

for small activity z. Mayer's conjecture, as formulated by Fisher in [F], *asserts that the function $p(z)$, defined by the power series and its analytic continuation, has, on the positive real axis, a nearest singularity $z = z_1$ which occurs at the condensation point $z = z_\sigma$.* One of our results is that *for short-range interaction potentials*, at least for a large class of models, *this form of Mayer's conjecture is correct at low temperature. The pressure cannot be analytically continued beyond $z = z_\sigma$, so that $z_1 = z_\sigma$.*

We cannot accept today all conclusions reached in [Ma]. However, from this paper and subsequent papers, the following understanding emerged about the question whether one can prove that, at sufficiently low temperatures, an isotherm consists of at least three different analytic parts. The picture is very different from the one in van der Waals' theory. This is well expressed in the introduction of [KaU].
(1) The equation for an isotherm is derived solely from the partition function.
(2) In order to have three different analytic parts for the isotherm one must take the thermodynamic limit[20].
(3) In the thermodynamic limit one cannot obtain states corresponding to supersaturated vapour states for example. Only equilibrium states are obtained.

[18]Today we know that this is not correct.

[19][F] is an excellent paper on this subject. See also [dB1], [dB2] and the foreword of Uhlenbeck in [Ka]. For an account of Mayer's theory, see [MaMa].

[20]This was emphasized by Kramers at the Van der Waals Centenary Congress, see [D]. He pointed out that one is really interested not in the partition function itself, but in the thermodynamic limit of the free energy. In this limit one may obtain non-analytic behaviour at certain densities and temperatures. However, it is the work of Yang and Lee [YLe], which established clearly this fact. It is true that in his famous paper [On] Onsager proved that the free energy of the two-dimensional Ising model, in the thermodynamic limit, has a singularity in the temperature, at zero magnetic field. But, this singularity is related to the critical point of the model, and is not the singularity studied in these lectures.

Kramers' statement about the thermodynamic limit should be taken cum grano salis. One can state, as basic principle of statistical equilibrium thermodynamics: *The partition function for finite systems is the basic object.* All equilibrium information about the system concerning bulk properties, like here, but also about surface properties of the system, as for example the wetting phenomenon [PfV], are encoded in the partition function. Since the number of particles is very large, the bulk properties of the system are best described in the thermodynamic limit. By considering the free energy in this limit, one singles out the bulk properties of the system. It should be stressed again, that the validity of this principle is mainly due to many very successful applications to a great variety of cases.

These very clear statements were not mathematically demonstrated[21] when they were formulated by Kahn and Uhlenbeck. In 1949 van Hove [vH] proved the existence of the thermodynamic limit and convexity properties of the thermodynamic potentials, which implies that the states on a isotherm correspond only to equilibrium states. For later mathematical works on this important question see [Ru]. In a famous paper [YLe] Yang and Lee demonstrated in 1952 how in the thermodynamic limit one can obtain singularities of the pressure, by accumulation on the real axis of complex zeros of the partition functions. In another paper [LeY] they illustrated this mathematical mechanism for the Ising model. It is, of course, one thing to prove the existence of a thermodynamic limit for the partition function describing both phases of a phase transition, but quite another thing to find out the exact nature of the discontinuity at the phase transition. These papers *do not contain any information about the nature of the singularity* of the pressure at a first order transition[22]. The question whether one can have an analytic continuation at a first order phase transition point is left open.

The mathematical deduction of the existence of a phase transition and of its properties, from the study of the partition function only, is very difficult for realistic models of physical systems. Motivated by the work of Mayer [Ma], several people, Bijl [Bij] (Leiden dissertation), Band [Ban], Frenkel [Fre1], [Fre2] and Mayer and Streeter [MaSt], introduced[23] the *droplet model*, in order to give a crude, but simple theory of condensation, which leads qualitatively to results comparable with those of Mayer's theory. Contrary to the work initiated in [Ma] this is a half-thermodynamics, half-statistics theory. The title of [Fre2] is *A General Theory of Heterophase Fluctuations and Pretransition Phenomena*, and in the abstract one reads: *[The paper] is based on the idea that the macroscopic transition of a substance from a phase A to a phase B is preceded by the formation of small nuclei being treated as resulting from "heterophase" density fluctuations or as manifestations of a generalized statistical equilibrium in which they play the roles of dissolved particles, whereas the A phase can be considered as the solvent.* The heterophase or heterogeneous fluctuations should be contrasted with the ordinary density fluctuations, which can be denoted as homophase or homogeneous fluctuations. The gaseous state is composed of single particles and of molecules or droplets containing several particles. The state of the system is specified by the number m_k of molecules with k particles, $k \geq 1$. One assumes that the interaction between droplets is negligible, and one postulates the form of the free energy of a droplet of l particles, which is a sum of two terms, one proportional to l (volume term) and another one proportional to $l^{\frac{2}{3}}$, representing a

[21] See the quotation of Siegert's paper [Si] in footnote 26.

[22] Chapter 15 of [Hu] (German edition (1964)) is an excellent exposition of these fundamental results obtained by van Hove and Yang and Lee. See also chapter two of [UF]. The main result in [YLe] is that, if a region of the complex plane is free of zeros of the partition functions, then the pressure is analytic in that region. Accumulation of the zeros of the partition functions is a necessary, but not sufficient condition for the existence of a singularity of the pressure. See [Sh] for examples of accumulation of the zeros on some points of the real axis, without producing a singularity of the pressure. For the mean-field Ising model there is accumulation of the zeros of the partition functions at $h = 0$, when the temperature is low enough, since the pressure is not analytic in the thermodynamic limit. But in this case, contrary to the theorem of Isakov, theorem 1.2, which is the main subject of these lectures, there is an analytic continuation of the pressure at $h = 0$ (see section 5).

[23] See in particular [F] and [dB1] for a treatment of this model.

boundary term, the surface free energy. This last term makes sense only for very large droplets. However, it is the behaviour of the statistics of large droplets which is important for the condensation phenomenon[24].

1.3. **The van der Waals limit.** A remarkable achievement of mathematical physics is the derivation of the van der Waals-Maxwell isotherms from statistical mechanics in the limiting case of infinitely long-range and infinitely weak interactions[25]. Brout in [Bro] studied the Ising model in this limit, in relation with the mean-field theory. He tried to develop a perturbation around the mean-field limit. He showed how one can recover this limiting case by taking the limit of infinitely long-range and infinitely weak interactions, so that the overall strength of the interaction is constant[26].

[24]The premises of this model are very different from those of a mean field approach. There are similarities and also important differences with the Pirogov-Sinai theory of phase transitions [PiSi]. In this theory the mechanism for phase coexistence is the following one. I consider only the case where the system has two ground-states, which are denoted by ψ_1, respectively ψ_2. In this theory, for sufficiently small temperatures, there are only two stable phases, called ψ_1-phase, respectively ψ_2-phase. Generically only one phase is stable, except for specific values of the parameters of the model.

Local perturbations of a ground-state are described by "geometric objects", called contours. (Usually the contours describing perturbations of the ground-state ψ_1 differ from those of the ground-state ψ_2.) A contour has a size, which corresponds to the region where the perturbation of the ground-state occurs. If the ψ_1-phase is the only stable phase, then *all* ψ_1-contours are stable (in a precise mathematical sense), while large ψ_2-contours are not stable. Stability of all ψ_1-contours (which is equivalent to say that the ψ_1-phase is stable) implies that the ground-state is stable with respect to local perturbations of any size, i.e. the ground-state for the infinitely extended system is stable. This is the origin of the ψ_1-phase.

More generally, if one considers a given region $\mathcal{R}$ of the ground-state ψ_j, then this region of the ground-state ψ_j is stable if and only if all ψ_j-contours inside $\mathcal{R}$ are stable. Notice that inside any given region $\mathcal{R}$ all possible perturbations occur with *non-zero* probability. The only way to stabilize a region $\mathcal{R}$ of the ground-state ψ_j, when all ψ_j-contours inside $\mathcal{R}$ are not stable is to *suppress* the unstable contours. This is precisely the basic idea of Zahradník in his fundamental paper [Z] about the Pirogov-Sinai theory.

An important difference with respect to the droplet model is that contours contain contours in their interiors, typically ψ_1-contours contains in their interiors ψ_2-contours and vice-versa. The stable ψ_2-contours are precisely those that give rise to droplets of phase ψ_2 inside the ψ_1-phase. Larger stable ψ_2-contours allows larger regions of the ground-state ψ_2-phase to become stable, and thus the appearance of larger droplets of the ψ_2-phase inside the ψ_1-phase. As one approaches a point of coexistence with the other phase (associated with ground-state ψ_2), more and more ψ_2-contours become stable. It is precisely, when *all* contours of both phases become stable that there is coexistence of the two phases. This happens at well-defined values of temperature and chemical potential. The system "knows" when condensation takes place. The stability of contours is a consequence of a delicate balance between volume versus surface effects. The subtle question of non-existence of an analytic continuation of the pressure at a first order phase transition point is also related to the stability/instability properties of the contours of both phases in a (complex) neighbourhood of the coexistence point.

I thank R. Fernández for helpful comments on the Pirogov-Sinai theory.

[25]Systems with weak long-range potentials are reviewed in [HLeb]. See also [Leb].

[26]From [S]: *The attitude of physicists toward the van der Waals equation has changed several times since its birth in 1873. After the Ursell-Mayer expansion in the mid-thirties, the orthodox view was that the van der Waals approximation was merely an extrapolation from the first two terms of the virial series, and the equal area construction an ex post facto introduction of thermodynamics, which would not be necessary if one could actually evaluate the partition function exactly, and obtain from it the pressure in the thermodynamic limit (number of particles $N \to \infty$, volume $V \to \infty$, with $N/V = v$ fixed).*

Baker also studied a similar limiting case for a one-dimensional spin system, [Ba] . However, the derivation of van der Waals' equation in this limit is due to Kac, Hemmer and Uhlenbeck in [KUH1], [KUH3] and [KUH3] for a one-dimensional model of N particles in an interval of length L, with hard-core of size $\delta > 0$ and interacting via an attractive interaction

$$-a\gamma \mathrm{e}^{-\gamma r}\,.$$

This model was introduced previously by Kac [K], who showed that the thermodynamic limit $L \to \infty$, $N \to \infty$ with $l = L/N$ constant, can be computed exactly. The free energy is equal, in this limit, to the maximum eigenvalue of a Hilbert-Schmidt kernel. For finite γ the model does not exhibit a phase transition, since it is a one-dimensional model with exponentially decaying interaction. However, if one takes the limit γ tending to 0, so-called *the van der Waals limit*, after the thermodynamic limit, then appears a phase transition, which is described by van der Waals' equation

$$\Big(p + \frac{a}{l^2}\Big)\Big(l - \delta\Big) = kT\,.$$

In 1964 van Kampen gave a derivation of van der Waals's equation with Maxwell's rule [vK]. The arguments of van Kampen are "local mean field" type arguments. This is the main and important difference with respect to a mean field theory, in which the state of the system is described by one real quantity, the density. The basic idea is that there are two scales. The system is divided into large cells, which are small compared to the range of the attractive interaction, but large enough in order to contain many of particles, and such that inside a cell one can use a mean-field approximation. In this way Van Kampen obtained a coarse-grained description of the model. The distribution of the particles is uniform over a cell, but not over the whole system. The system can be partly in a gaseous phase or partly in a liquid phase, and one can define a free energy for a given non-homogeneous coarse-grained distribution, which is essentially the sum of the free energies of the cells. The equilibrium free energy of the whole system is obtained by minimizing the free energy over non-homogeneous coarse-grained distributions.

Lebowitz and Penrose [LebP], combining the ideas of van Kampen and the van der Waals limit, proved the following remarkable result. Let $\varsigma : \mathbb{R}^d \to \mathbb{R}$, $\varsigma(x) = \varsigma(|x|)$ be a (positive) function with compact support in $[-1,1]^d$, so that

$$\int \varsigma(x)\, dx = \alpha > 0\,.$$

There was, however, at that time no proof for any classical system that the van der Waals loops would not result even from an exact calculation of the Gibbs integral and that phase transitions were contained in the fundamental formalism of equilibrium statistical mechanics; and some very respectable physicists expressed a minority view to the contrary still in the late thirties.

In the forties, however, the famous papers of Onsager and of van Hove appeared. [...]. *The latter proved the impossibility of van der Waals loops in the thermodynamic limit for systems of particles with hard-core repulsive interaction and finite range interaction.* [...].

All this seemed to close the case, but I remember Professor Uhlenbeck saying at that time, that is in the late forties, that he was unwilling to believe that a theory which gave a qualitative and semi-quantitative description of the isotherm, including the phase transition, did not have some nucleus of truth in it.

The idea that this nucleus of truth would be a limit theorem stating that that the mean field approximation should be exact *in the limit of infinitely weak interaction of infinite range, must have been in people's minds; but I think it was first put in print by Brout in his 1960 Ising model paper, in which he also started the investigation of the neighbourhood of this limit.*

Let $0 < \gamma < 1$. The interaction potential between particles located at $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$ is given by

$$\phi_\gamma(|x-y|) = q(|x-y|) - \gamma^d \varsigma(\gamma|x-y|)\,,$$

where $q(|x|)$ is a fixed short-range repulsive potential which diverges at the origin more rapidly than $|x|^{-d'}$, $d' > d$. If the interaction potential is q only (reference system), then the free energy (at given temperature and in the thermodynamic limit) is $\tilde{f}(\rho)$, whereas the free energy (in the thermodynamic limit) for the full interaction potential ϕ_γ is denoted by $f_\gamma(\rho)$. By general results $f_\gamma(\rho)$ is convex. Therefore, as one takes the van der Waals limit $\gamma \to 0$, the limiting free energy remains convex. However, this limiting convex free energy is the convex envelope of the non-convex free energy

$$-\frac{1}{2}\alpha\,\rho^2 + \tilde{f}(\rho)\,.$$

Theorem 1.1. *Under the above hypothesis, in the van der Waals limit $\gamma \to 0$,*

$$\lim_{\gamma\to 0} f_\gamma(\rho) = \mathrm{CE}\Big[-\frac{1}{2}\alpha\,\rho^2 + \tilde{f}(\rho)\Big]\,,$$

where $\mathrm{CE}[g]$ means the convex envelope of the function g.

1.4. **Droplet models versus mean field models.** Since the problem of the nature of the singularity at a first order phase transition is difficult and subtle, it was discussed in the 50's and 60's by making drastic assumptions. Two different approaches were considered, one based on mean field type assumptions, and the other on the droplet model. Both could not give an answer to the question, but allowed to formulate at least precise conjectures.

Temperley, [T], and Katsura, [Ka1] and [Ka2], considered the Bragg-Williams approximation of lattice gases, in which the spatial positions of the particles do not play any role. They observed a kink in the graph of the pressure at some activity z_{YL}, but showed that the virial expansion had a singularity at z^* strictly larger than z_{YL}, which implies that Mayer's conjecture is false and that the pressure has an analytic continuation beyond z_{YL}, as in the van der Waals-Maxwell Theory. Katsura conjectured that this is also true for simple models with finite range interaction, like the Ising model.

A completely different conclusion follows from an analysis of the droplet model [A], [F] and [La1][27]. This model, as opposed to the mean field approximation, predicts that the finiteness of the range of interaction plays a crucial role in the analytic properties of the thermodynamic potentials. Namely, when the range of interaction is finite, droplets of any size are stable at the condensation point, and although the probability of occurrence of *large* droplets is very small, it is their stability which yields a contribution of the order $k!^{\frac{d}{d-1}}$ to the k-th derivative of the pressure, and which prevents an analytic continuation. At the transition point $z = z_\sigma$ all derivatives with respect to z (for real z) remain finite; one can write a Taylor series for the pressure p, but this Taylor series has a convergence radius equal to 0, so that no analytical continuation of p across z_σ is possible.

[27] The conclusion that the condensation point should be an essential singularity of the activity series was advanced at the I.U.P.A.P. Conference on Phase Transitions at Brown University in June 1962 by Fisher. See footnote 17 in [F].

The question of the possibility of an analytic continuation was analyzed afterwards by several people. Using different techniques (exact computation, spectral properties of transfer matrix, numerical analysis, series expansion methods) they obtained different answers and no definitive conclusions. The only definitive result was obtained in 1978 by Kunz and Souillard [KuSo], who proved that the generating function of the cluster-size distribution in percolation is analytic at $z = 0$ in absence of percolation, and has, in the percolation regime, a singularity at $z = 0$, which is of the same kind as the one of the pressure of the droplet model. The importance of this paper is that it is the first mathematical result on this difficult question, which is free from any assumptions about the behaviour of model. However, the problem solved by Kunz and Souillard is mathematically closer to the droplet model than to the Ising model.

The breakthrough came with the profound work of Isakov [I1] in 1984.

Theorem 1.2 (Isakov)**.** *In dimension $d \geq 2$, at low enough temperature, the pressure*[28] *of the Ising model in a magnetic field h, $p = p(h)$, is infinitely differentiable at $h = 0^{\pm}$, and for large k*

$$p^{(k)}(0^{\pm}) \sim C^k k!^{\frac{d}{d-1}} .$$

The result implies that one can define two Taylor series of the pressure at $h = 0$ by evaluating the derivatives at $h = 0^+$, respectively $h = 0^-$. Both series have zero convergence radius, so that there is no analytic continuation of p from $\{h < 0\}$ to $\{h > 0\}$ across $h = 0$, or vice versa. In a second paper [I2], Isakov tried to extend this result to generic two phase lattice models. He had, however, to introduce hypotheses that are not easy to verify in concrete models. The analysis of Isakov confirms the prediction of the droplet model. However, as already mentioned in the footnote 24 the mechanism in lattice models leading to phase coexistence and non-analytic continuation of the pressure is much more subtle than in the droplet model.

[28] The terminology here is not the usual terminology, when the model is considered as a spin model. The term "pressure" is used in general for the lattice gas interpretation of the model. It is the grand canonical pressure. In that case the chemical potential μ is related to the magnetic field by

$$\mu = 2h - 4J \quad (J, \text{ coupling constant of the model}) .$$

Then the transition takes place at $\mu^* = -4J$. In the rest of the lectures I adopt the spin formulation of the model, but I use the term "pressure" instead of "free energy".

2. Absence of analytic continuation for lattice models with short-range interaction

At a first order phase transition, the pressure does not have an analytic continuation in the thermodynamic variable, which is conjugate to an order parameter for the transition, [FrPf1]. I give a precise statement of this result, theorem 2.1, in the framework of the Pirogov-Sinai theory, at low temperatures, for lattice models with finite range interaction and two periodic ground-states, under the only condition that the Peierls condition is verified.

2.1. **Main result, theorem 2.1.** The notations are close to those of [FrPf1]. As it is usually the case, the models are defined for the cubic lattice

$$\mathbb{Z}^d := \{x = (x(1), \dots, x(d)) : x(i) \in \mathbb{Z}\} \quad \text{with} \quad d \geq 2\,,$$

which is equipped with a norm,

$$|x| := \max_{i=1}^{d} |x(i)|\,.$$

There is a natural notion of "translation by a" for all $a \in \mathbb{Z}^d$. For finite $R > 0$,

$$B_R(x) := \{y \in \mathbb{Z}^d : |x - y| \leq R\}\,.$$

At each site of the lattice there is a "spin" taking its values in S, the state space of the model, which is a finite subset. A configuration of the system is a function $\varphi : \mathbb{Z}^d \to S$. The restriction of φ to $A \subset \mathbb{Z}^d$ is denoted by $\varphi(A)$, and two configurations φ, ψ are almost surely equal, $\varphi = \psi$ (a.s.), if $\{x : \varphi(x) \neq \psi(x)\} \subset \mathbb{Z}^d$ is finite.

The interaction between spins are defined by a potential[29], which is a family $\{\Phi_A\}$ of local maps indexed by finite subsets of $\mathbb{Z}^d$,

$$\varphi \mapsto \Phi_A(\varphi) \in \mathbb{R} \quad \Phi_A(\varphi) = \Phi_A(\psi), \text{ whenever } \varphi(A) = \psi(A).$$

The interaction is of finite range, if there exists $R < \infty$ such that

$$\Phi_A \equiv 0 \text{ if } \nexists\, a \in \mathbb{Z}^d \text{ such that } A \subset B_R(a).$$

It is convenient to introduce

$$\mathcal{U}_x := \sum_{A \ni x} \frac{1}{|A|} \Phi_A\,,$$

and to write a hamiltonian $\mathcal{H}$ as the formal sum $\mathcal{H} = \sum_{x \in \mathbb{Z}^d} \mathcal{U}_x$. The partition function[30] of the spins indexed by $x \in \Lambda \subset \mathbb{Z}^d$, at inverse temperature β, is

$$Z(\Lambda; \beta) := \sum_{\varphi(\Lambda)} \exp\Big(-\beta \sum_{A \subset \Lambda} \Phi_A(\varphi(\Lambda))\Big)\,,$$

and the pressure, in the thermodynamic limit, is

$$p(\beta) := \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{\beta|\Lambda|} \ln Z(\Lambda; \beta) \quad \text{(van Hove limit).}$$

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[29] See [Ru] for further details.

[30] One can add boundary terms. In the thermodynamic limit, taken in the sense of van Hove (see [Ru]), these boundary terms do not affect the pressure.

Let $\mathcal{H}_0$ be a hamiltonian with interaction of finite range R,

$$\mathcal{H}_0 = \sum_{x \in \mathbb{Z}^d} \mathcal{U}_{0,x} \,.$$

I assume that $\mathcal{H}_0$ has two (periodic) ground-states, ψ_1 and ψ_2. A configuration ψ_1 is a ground-state if

$$\mathcal{H}_0(\varphi|\psi_1) := \sum_{x \in \mathbb{Z}^d} \big(\mathcal{U}_{0,x}(\varphi) - \mathcal{U}_{0,x}(\psi_1)\big) \geq 0 \quad \text{for any } \varphi = \psi_1 \text{ (a.s.)} \,.$$

$\mathcal{H}_0(\varphi|\psi_1)$ is well-defined since the sum has only finitely many non-zero terms. Given φ, a lattice site x is ψ_j-correct if

$$\varphi(B_R(x)) = \psi_j(B_R(x)) \,.$$

It is correct if it is ψ_1-correct or ψ_2-correct, otherwise it is incorrect. The boundary of a configuration φ is by definition the subset of $\mathbb{Z}^d$

$$\partial\varphi := \bigcup_{\substack{x \in \mathbb{Z}^d :\, x \\ \text{incorrect for } \varphi}} B_R(x) \,.$$

Main assumption. *The two ground-states ψ_m of $\mathcal{H}_0$, $m = 1, 2$, are periodic. They verify* the Peierls condition*: there exists a constant $\rho > 0$ such that*

$$\mathcal{H}_0(\varphi|\psi_m) \geq \rho|\partial\varphi| \quad \forall\, \varphi \text{ such that } \varphi = \psi_m \text{ (a.s.)} \,.$$

$|C|$ denotes the cardinality of a finite subset C.

The Peierls condition is a very natural assumption. It means that in order to create a boundary one needs an energy at least proportional to the size of the boundary. Boundaries are energy barriers.

Let $\mathcal{H}_1$ be another hamiltonian with interaction of finite range R,

$$\mathcal{H}_1(\varphi) = \sum_{x \in \mathbb{Z}^d} \mathcal{U}_{1,x} \,.$$

The hamiltonian of the model, $\mathcal{H}^\mu$, is the sum of $\mathcal{H}_0$ and $\mu\mathcal{H}_1$,

$$\mathcal{H}^\mu := \mathcal{H}_0 + \mu\mathcal{H}_1 \,, \quad \mu \in \mathbb{R} \,.$$

Assumption. $\mathcal{H}_1$ splits the degeneracy of the ground-states of $\mathcal{H}_0$*: if $\mu < 0$, $\mathcal{H}^\mu$ has a unique ground-state, which is ψ_2; if $\mu > 0$, $\mathcal{H}^\mu$ has a unique ground-state, which is ψ_1.*

To simplify slightly the exposition[31], I further assume that the energy (per spin) of the ground-states for the hamiltonian $\mathcal{H}_0$ is given by

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \mathcal{U}_{0,x}(\psi_m) = \mathcal{U}_{0,y}(\psi_m) = 0 \,, \quad \forall\, y \,,\ m = 1, 2 \,.$$

Similarly, the energy (per spin) of ψ_m for the hamiltonian $\mathcal{H}_1$ is

$$h(\psi_m) := \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \mathcal{U}_{1,x}(\psi_m) = \mathcal{U}_{1,y}(\psi_m) \,, \quad \forall\, y \,,\ m = 1, 2 \,.$$

[31] See the computation in (3.4). This is not a genuine restriction, since one can always, by an appropriate change of the lattice and of the state space S, reduce the general case to the case considered in these lectures.

The assumption, about the splitting of the ground-states of $\mathcal{H}_0$ by $\mathcal{H}_1$, implies that

$$\Delta := h(\psi_2) - h(\psi_1) > 0\,.$$

The quantity $\mathcal{U}_{1,x}$ is interpreted as an order parameter. The real quantity μ is the conjugate variable to this order parameter. The pressure of the model, at inverse temperature β, is written by $p(\mu, \beta)$.

Theorem 2.1. *Under the above setting, there exist an open interval $U_0 \ni 0$, $\beta^* \in \mathbb{R}^+$ and, for all $\beta \geq \beta^*$ there exists a $\mu^*(\beta) \in U_0$ with the following properties.*

(1) *There is a first-order phase transition at $\mu^*(\beta)$.*
(2) *The pressure $p(\mu, \beta)$ is real-analytic in μ in $\{\mu \in U_0 : \mu < \mu^*(\beta)\}$; it has a C^∞ continuation in $\{\mu \in U_0 : \mu \leq \mu^*(\beta)\}$.*
(3) *The pressure $p(\mu, \beta)$ is real-analytic in μ in $\{\mu \in U_0 : \mu > \mu^*(\beta)\}$; it has a C^∞ continuation in $\{\mu \in U_0 : \mu \geq \mu^*(\beta)\}$.*
(4) *There is no analytic continuation of p along a real path from $\mu < \mu^*(\beta)$ to $\mu > \mu^*(\beta)$ crossing $\mu^*(\beta)$, or vice-versa.*

2.2. **Comments on theorem 2.1.** These results generalize the works of Isakov [I1], for the Ising model, and [I2], where a similar theorem is proven under additional assumptions, which are not easy to verify in a concrete model. Theorem 2.1, which relies only on the Peierls condition, is therefore a genuine improvement of [I2]. The first statement is a particular case of the theory of Pirogov and Sinai, see [PiSi] and [Si]. I give a proof of this result in section 3, as far as it concerns the pressure, since one needs detailed information about the phase diagram in the complex plane of the parameter μ.

The obstruction to an analytic continuation of the pressure in the variable μ is due to the stability of the contours of both phases in a neighborhood of μ^*. The proof follows in essence that of Isakov in [I1]. It is given in section 4.

The results presented here are true for a much larger class of systems. For example, for the Potts model with high number q of components at the first order phase transition point β_c, where the q ordered phases coexist with the disordered phase. Here β, the inverse temperature, plays the role of the field μ, and the statement is that the pressure, which is analytic for $\beta > \beta_c$, or for $\beta < \beta_c$, does not have an analytic continuation across β_c. They are also true when the model has more than two ground-states. For example, for the Blume-Capel model, whose hamiltonian is

$$\sum_{x,y}(s_x - s_y)^2 - h\sum_x s_x - \lambda\sum_x s_x^2 \quad \text{with} \quad s_x \in \{-1, 0, 1\}\,,$$

the pressure is an analytic function of h and λ in the single phase regions. At low temperature, at the triple point occurring at $h = 0$ and $\lambda = \lambda^*(\beta)$ there is no analytic continuation of the pressure in λ, along the path $h = 0$, or in the variable h, along the path $\lambda = \lambda^*$ [FrPf4].

3. The theory of phase transitions of Pirogov and Sinai

Since theorem 2.1 is proved in the framework of the theory of phase transitions of Pirogov and Sinai, the first step is to write the model as a contour model. The reader is invited to read the footnote 24, in which the basic mechanism of phase condensation for lattice models is exposed without any technicalities. The idea of a contour model is to obtain a representation of the partition function $\Theta_q(\Lambda)$ (definition 3.4) in terms of geometric objects, the contours, which interact only through a hard-core condition. I present the Pirogov-Sinai theory as a perturbative theory around $\beta = \infty$. At $\beta = \infty$ the phase transition takes place at $\mu = 0$, where there is coexistence of the two ground-states. The approach, which I expose below, consists in constructing the phase transition point perturbatively by taking into account in a systematic manner the different perturbations, "excitations", of the model, and by focusing the attention to the phase coexistence point, starting from the point $\mu = 0$ where there is coexistence of the two ground-states. In an interval I_n of $\mu = 0$, when β is large, but finite, one can defined constrained pressures, p_1^n and p_2^n, for both phases, by taking into account only finitely many different kinds of contours. The constrained pressure p_q^n is *analytic in* I_n. One defines the transition point in the interval I_n by finding the value μ^*_{n+1} of μ such that

$$p_1^n(\mu^*_{n+1}, \beta) = p_2^n(\mu^*_{n+1}, \beta)\,.$$

$I_{n+1} \subset I_n$, and as n increases, the length of the interval tends to zero. This determines uniquely a point μ^* where *all* contours are stable. This is the phase coexistence point.

This approach is particularly well adapted since it can be done also for *complex* values of the parameter μ, which is an essential point for examining the nature of the singularity of the free energy at μ^*. This is the original method[32] of Isakov in [I2], who, for the first time constructed phase diagrams in the complex μ-plane. It differs from that of [PiSi], which is based on the Banach fixed-point theorem.

3.1. **Lattice models as contour models.**

I follow the text of Sinai [Si]. Further details may be found in that reference.

Definition 3.1. *Let M denote a finite connected* [33] *subset of $\mathbb{Z}^d$, and φ a configuration. A couple $\Gamma = (M, \varphi(M))$ is called a* contour of the configuration *φ if M is a component of the boundary $\partial\varphi$. A couple $\Gamma = (M, \varphi(M))$ is a* contour *if there exists a configuration such that Γ is a contour of that configuration.*

---

[32] In [Z] another approach is developed, which has similar features, and which has been used by many people. Zahradník defines, by brute force, i.e. by suppressing unstable contours, truncated pressures for both phases on the whole phase diagram. So, for each value of μ, one has two different truncated pressures, and the equilibrium pressure of the model is equal to the maximal (with my definition of pressure) truncated pressure, so that the transition point is given by the value of μ for which the two truncated pressures are equal. Technically this approach is more involved, if one wants to construct smooth truncated pressures. (In the original paper the truncated pressures are not even continuous.) This approach works well for complex values of μ, see [BorIm]. Since the truncated pressures cannot be analytic in general, they are inappropriate in the present context.

[33] A path on $\mathbb{Z}^d$ is a set of points $\{x_0, x_1, \ldots, x_n\}$ with the property that $|x_i - x_{i-1}| = 1$ for all $i = 1, \ldots, n$.
Connected set means path-connected set, and a component B of a subset $A \subset \mathbb{Z}^d$ is a maximally path-connected subset of A.

The subset M of $\Gamma = (M, \varphi(M))$ is the support of the contour, and is denoted by $\operatorname{supp}\Gamma$, or simply by Γ when no confusion arises. In particular

$$|\Gamma| \equiv |\operatorname{supp}\Gamma| .$$

Let A_α be the components of $\mathbb{Z}^d \backslash M$. For each component A_α there exists a unique label $q(\alpha) \in \{1,2\}$ such that

$$\varphi_\Gamma(x) := \begin{cases} \psi_{q(\alpha)}(x) & \text{if } x \in A_\alpha \\ \varphi(x) & \text{if } x \in M \end{cases}$$

is the unique configuration with the property that $\partial\varphi_\Gamma = M$ and $\varphi_\Gamma(M) = \varphi(M)$. There is only one infinite component A_α, called exterior of Γ, which is denoted by $\operatorname{Ext}\Gamma$. All other components are the internal components; $\operatorname{Int}_m \Gamma$ is the union of all internal components of Γ with label m; the interior of Γ is $\operatorname{Int}\Gamma := \bigcup_{m=1,2} \operatorname{Int}_m \Gamma$. In order to indicate the label of $\operatorname{Ext}\Gamma$, a superscript is added to Γ. Thus, Γ^q means that on $\operatorname{Ext}\Gamma$ the configuration φ_Γ is equal to the ground-state configuration ψ_q. Γ^q is a contour with boundary condition ψ_q. By definition, the volume of a contour Γ^q, with boundary condition ψ_q, is the total volume of the internal components of Γ^q with label m, $m \neq q$:

$$V(\Gamma^q) := |\operatorname{Int}_m \Gamma^q| \quad (m \neq q) .$$

Definition 3.2. *Let $\Lambda \subset \mathbb{Z}^d$. A contour Γ is* inside Λ, *which is written $\Gamma \subset \Lambda$, if $\operatorname{supp}\Gamma \subset \Lambda$, $\operatorname{Int}\Gamma \subset \Lambda$ and* [34] *$d(\operatorname{supp}\Gamma, \Lambda^c) > 1$. A contour Γ of a configuration φ is an* external contour of φ *if $\operatorname{supp}\Gamma \subset \operatorname{Ext}\Gamma'$ for any other contour Γ' of φ. A* compatible family of contours in Λ *is a family of contours with the same boundary condition, say $\{\Gamma_1^q, \dots, \Gamma_n^q\}$, with $\Gamma_i^q \subset \Lambda$ and $d(\operatorname{supp}\Gamma_i^q, \operatorname{supp}\Gamma_j^q) > 1$ for all $i \neq j$.*

The basic statistical mechanical quantities of the theory are

(1) the partition function $\Theta(\Gamma^q)$ of the contour Γ^q,

(2) the partition function $\Theta_q(\Lambda)$ of the system in Λ, with boundary condition ψ_q,

(3) the weight $\omega(\Gamma^q)$ of the contour Γ^q.

Definition 3.3. *Let $\Omega(\Gamma^q)$ be the set of configurations $\varphi = \psi_q$ (a.s.) such that Γ^q is the only external contour of φ. The* partition function of Γ^q *is*

$$\Theta(\Gamma^q) := \sum_{\varphi \in \Omega(\Gamma^q)} \exp\big[-\beta\mathcal{H}(\varphi|\psi_q)\big] .$$

Definition 3.4. *Let $\Omega_q(\Lambda)$ be the set of configurations $\varphi = \psi_q$ (a.s.) such that $\Gamma \subset \Lambda$ whenever Γ is a contour of φ. The* partition function of the system in Λ, with boundary condition ψ_q, *is*

$$\Theta_q(\Lambda) := \sum_{\varphi \in \Omega_q(\Lambda)} \exp\big[-\beta\mathcal{H}(\varphi|\psi_q)\big] .$$

Definition 3.5. *Let Γ^q be a contour with boundary condition ψ_q. The* weight $\omega(\Gamma^q)$ of Γ^q *is*

$$\omega(\Gamma^q) := \exp\big[-\beta\mathcal{H}(\varphi_{\Gamma^q}|\psi_q)\big] \frac{\Theta_m(\operatorname{Int}_m \Gamma^q)}{\Theta_q(\operatorname{Int}_m \Gamma^q)} \quad (m \neq q) .$$

The (bare) surface energy of a contour Γ^q *is*

$$\|\Gamma^q\| := \mathcal{H}_0(\varphi_{\Gamma^q}|\psi_q) .$$

[34] If $A \subset \mathbb{Z}^d$, $B \subset \mathbb{Z}^d$, then $d(A,B) := \min_{x\in A} \min_{y \in B} |x-y|$.

For each ground-state ψ_q one defines a ψ_q-dependent pressure (limit in the sense of van Hove)

$$g_q := \lim_{\Lambda\uparrow\mathbb{Z}^d} \frac{1}{\beta|\Lambda|} \ln \Theta_q(\Lambda)\,.$$

It is easy to verify that the pressure of the model, in the thermodynamic limit, verifies

$$g_q = p + \mu\, h(\psi_q)\,.$$

It does not depend on ψ_q, contrary to g_q. The partition function $\Theta_q(\Lambda)$ is equal to

$$\Theta_q(\Lambda) = \sum \prod_{i=1}^{n} \Theta(\Gamma_i^q)\,, \tag{3.1}$$

where the sum is over the set of all compatible families $\{\Gamma_1^q, \ldots, \Gamma_n^q\}$ of external contours in Λ. On the other hand

$$\Theta(\Gamma^q) = \exp\big[-\beta\mathcal{H}(\varphi_{\Gamma^q}|\psi_q)\big] \prod_{m=1}^{2} \Theta_m(\mathrm{Int}_m\, \Gamma^q)\,. \tag{3.2}$$

Replacing $\Theta(\Gamma_i^q)$ in (3.1) by its expression given by (3.2), taking into account definition 3.5, and iterating this procedure, one obtains easily the final form of the partition function $\Theta_q(\Lambda)$, as the partition function of a contour model, i.e.

$$\Theta_q(\Lambda) = 1 + \sum \prod_{i=1}^{n} \omega(\Gamma_i^q)\,, \tag{3.3}$$

the sum being over all compatible families of contours $\{\Gamma_1^q, \ldots, \Gamma_n^q\}$ with boundary condition ψ_q.

Let Γ^q be a contour and $m \neq q$.

$$\begin{aligned}
\mathcal{H}(\varphi_{\Gamma^q}|\psi_q) &= \sum_{x\in\mathbb{Z}^d} \big(\mathcal{U}_{0,x}(\varphi_{\Gamma^q}) + \mu\,\mathcal{U}_{1,x}(\varphi_{\Gamma^q}) - \mathcal{U}_{0,x}(\psi_q) - \mu\,\mathcal{U}_{1,x}(\psi_q)\big) \\
&= \mathcal{H}_0(\varphi_{\Gamma^q}|\psi_q) \\
&+ \sum_{x\in\mathrm{supp}\,\Gamma^q} \mu\big(\mathcal{U}_{1,x}(\varphi_{\Gamma^q}) - \mathcal{U}_{1,x}(\psi_q)\big) + \sum_{x\in\mathrm{Int}\,\Gamma^q} \mu\big(\mathcal{U}_{1,x}(\varphi_{\Gamma^q}) - \mathcal{U}_{1,x}(\psi_q)\big) \\
&= \|\Gamma^q\| + \mu \sum_{x\in\mathrm{supp}\,\Gamma^q} \big(\mathcal{U}_{1,x}(\varphi_{\Gamma^q}) - \mathcal{U}_{1,x}(\psi_q)\big) + \mu(h(\psi_m) - h(\psi_q))V(\Gamma^q) \\
&\equiv \|\Gamma^q\| + \mu a(\varphi_{\Gamma^q}) + \mu(h(\psi_m) - h(\psi_q))V(\Gamma^q)\,.
\end{aligned} \tag{3.4}$$

In (3.4)

$$a(\varphi_{\Gamma^q}) := \sum_{x\in\mathrm{supp}\,\Gamma^q} \mathcal{U}_{1,x}(\varphi_{\Gamma^q}) - \mathcal{U}_{1,x}(\psi_q)\,.$$

Since the interaction is bounded, there exists a constant C_1 so that

$$|a(\varphi_{\Gamma^q})| \leq C_1|\Gamma^q|\,. \tag{3.5}$$

The surface energy $\|\Gamma^q\|$ is always strictly positive since the Peierls condition holds, and there exists a constant C_2, independent of q, such that

$$\rho|\Gamma^q| \leq \|\Gamma^q\| \leq C_2|\Gamma^q|\,. \tag{3.6}$$

Definition 3.6. *The weight $\omega(\Gamma^q)$ is τ-stable for Γ^q if there exists $\tau > 0$ such that*

$$|\omega(\Gamma^q)| \leq \exp(-\tau|\Gamma^q|)\,.$$

The dominant terms of the weight $\omega(\Gamma^q)$, in the neighbourhood of $\mu = 0$, are $\|\Gamma^q\|$, the bare surface energy of Γ^q, and $\mu(h(\psi_m) - h(\psi_q))V(\Gamma^q)$, which is a volume term. Stability of the weight is true when surface terms dominate volume terms (see (3.4)). Therefore, in the proof of the stability of weights, the isoperimetric inequality

$$\chi_q V(\Gamma^q)^{\frac{d-1}{d}} \leq \|\Gamma^q\|$$

plays a central role. There is a complication, due to the fact that there is no homogeneity property for this inequality, and it is very difficult to determine the value of χ_q. The way how this constant is defined is an important point. The precise formulation of the isoperimetric inequality, which is convenient in the context of theorem 2.1 is defined later in (4.1). This is a key point of the analysis.

The construction of the phase diagram is done by considering constrained partition functions and constrained pressures involving only contours such that $V(\Gamma^q) \leq n$, $n \in \mathbb{N}$. The phase diagram is constructed for these constrained pressures, and then one takes the limit $n \to \infty$. For given n, $n = 0, 1, \ldots$, the weight $\omega_n(\Gamma^q)$ is defined by

$$\omega_n(\Gamma^q) := \begin{cases} \omega(\Gamma^q) & \text{if } V(\Gamma^q) \leq n, \\ 0 & \text{otherwise.} \end{cases} \tag{3.7}$$

Let $l(n)$ be defined on $\mathbb{N}$ by

$$l(n) := C_0^{-1} \lceil 2dn^{\frac{d-1}{d}} \rceil \quad n \geq 1\,.$$

This function has the property[35]:

$$V(\Gamma^q) \geq n \quad \Longrightarrow \quad |\Gamma^q| \geq l(n)\,.$$

So, if the volume $V(\Gamma^q)$ of a contour is large, then its surface energy cannot be too small (see (3.6)). For $q = 1, 2$, one defines constrained partition functions Θ_q^n by equation (3.3), using $\omega_n(\Gamma^q)$ instead of $\omega(\Gamma^q)$. It is essential to replace the real parameter μ by a complex parameter z; provided that $\Theta_q^n(\Lambda)(z) \neq 0$ for all Λ,

$$g_q^n(z) := \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{\beta|\Lambda|} \ln \Theta_q^n(\Lambda)(z) \quad \text{and} \quad p_q^n(z) := g_q^n(z) - z\,h(\psi_q)\,. \tag{3.8}$$

p_q^n is the constrained pressure of order n and boundary condition ψ_q. Contrary to p, it depends on the boundary condition.

Lemma 3.1 gives basic, but essential, estimates for the rest of the paper. The only hypothesis for this lemma is that the weights of the contours are τ-stable.

Lemma 3.1. *Let $\omega(\Gamma^q)$ be any complex weights, depending on a parameter t. The weight $\omega_n(\Gamma^q)$ is defined by (3.7).*
(A) Suppose that the weights $\omega_n(\Gamma^q)$ are τ-stable for all Γ^q, as well as the weights

[35] Given $\Lambda \subset \mathbb{Z}^d$, one defines $\partial|\Lambda|$ as the $(d-1)$-volume of the boundary of the set in $\mathbb{R}^d$ which is the union of unit cubes centered at the points of Λ. One has

$$2d|\Lambda|^{\frac{d-1}{d}} \leq \partial|\Lambda| \quad \text{(isoperimetric inequality)}\,.$$

The constant C_0 is such that, if $\Lambda = \mathrm{Int}_m\,\Gamma^q$ and $\partial V(\Gamma^q) := \partial|\Lambda|$, then

$$\partial V(\Gamma^q) \leq C_0|\Gamma^q|\,.$$

$\frac{d}{dt}\omega_n(\Gamma^q)$ and $\frac{d^2}{dt^2}\omega_n(\Gamma^q)$. Then there exists $K<\infty$ and $\tau^*<\infty$ independent of n, so that for all $\tau\geq\tau^*$,

$$\beta\Big|\frac{d^k}{dt^k}g_q^n\Big|\leq K\mathrm{e}^{-\tau}\quad k=0,1,2\,.$$

For all finite subsets $\Lambda\subset\mathbb{Z}^d$,

$$\Big|\frac{d^k}{dt^k}\ln\Theta_q^n(\Lambda)-\beta\frac{d^k}{dt^k}g_q^n\,|\Lambda|\Big|\leq K\mathrm{e}^{-\tau}\,\partial|\Lambda|\quad k=0,1,2\,.$$

(B) If $\omega_n(\Gamma^q)=0$ for all Γ^q such that $|\Gamma^q|\leq m$, then

$$\beta|g_q^n|\leq\big(K\mathrm{e}^{-\tau}\big)^m\,.$$

For $n\geq m$ and $m\geq 1$

$$\beta|g_q^n-g_q^{m-1}|\leq\big(K\mathrm{e}^{-\tau}\big)^{l(m)}\,.$$

(C) If the weights $\omega_n(\Gamma^q)$ are τ-stable for all Γ^q and all $n\geq 1$, then all these estimates hold for g_q and Θ_q instead of g_q^n and Θ_q^n. Moreover, $\frac{d^k}{dt^k}g_q^n$ converge to $\frac{d^k}{dt^k}g_q$ for $k=0,1,2$.

The proof of lemma 3.1 is based on the identity

$$\ln\Theta_q^n(\Lambda)=\sum_{m\geq1}\frac{1}{m!}\sum_{\Gamma_1^q\subset\Lambda}\cdots\sum_{\Gamma_m^q\subset\Lambda}\varphi_m^T(\Gamma_1^q,\ldots,\Gamma_m^q)\prod_{i=1}^m\omega_n(\Gamma_i^q)\,,\tag{3.9}$$

which is valid when the weights $\omega_n(\Gamma^q)$ of all contours with boundary condition ψ_q are τ-stable, and if τ is large enough. In (3.9) the convergence is absolute, and $\varphi_m^T(\Gamma_1^q,\ldots,\Gamma_m^q)$ is a purely combinatorial factor. This identity is studied in detail in [Pf]. Lemma 3.1 implies that

$$\frac{d^k}{dt^k}\ln\Theta_q^n(\Lambda)=\beta\frac{d^k}{dt^k}g_q^n\,|\Lambda|+\text{surface term}\quad k=0,1,2\,,$$

with a uniform control of $\frac{d^k}{dt^k}g_q^n$ and surface terms. Both terms are $O(\mathrm{e}^{-\tau})$. See [FrPf1] for a proof.

3.2. **Construction of the phase diagram in the complex z-plane.** The Pirogov-Sinai theory rests on few basic concepts: (a) the notion of contour, together with the notion of weight of contour, (b) the notion of stability of contour, (c) the Peierls condition. Technically, the basic formula is (3.4), which, together with the Peierls condition and lemma 3.1, allow to establish stability of a contour. In this subsection I carry out the program set forth at the beginning of section 3.

To construct the phase diagram for complex values of the parameter μ, one constructs iteratively the phase diagram for the constrained pressures p_q^n (see (3.8)). Set $z:=\mu+i\nu$. The method consists in finding a sequence of intervals for each $\nu\in\mathbb{R}$,

$$U_n(\nu;\beta):=\big(\mu_n^*(\nu;\beta)-b_n^1,\mu_n^*(\nu;\beta)+b_n^2\big)\,,$$

with the properties (1):

$$\big(\mu_n^*(\nu;\beta)-b_n^1,\mu_n^*(\nu;\beta)+b_n^2\big)\subset\big(\mu_{n-1}^*(\nu;\beta)-b_{n-1}^1,\mu_{n-1}^*(\nu;\beta)+b_{n-1}^2\big)\,,\tag{3.10}$$

and (2): $\lim_n b_n^q = 0$, $q = 1, 2$. On the intervals $U_{n-1}(\nu;\beta)$ the constrained pressures p_q^{n-1} of order $n-1$, $q = 1, 2$, are well-defined and analytic on

$$\mathbb{U}_{n-1} := \{z \in \mathbb{C} :\ \mathrm{Re} z \in U_{n-1}(\mathrm{Im} z;\beta)\}\,.$$

The point $\mu_n^*(\nu;\beta)$, $n \geq 1$, is the (unique) solution of the equation

$$\mathrm{Re}\big(p_2^{n-1}(\mu_n^*(\nu;\beta) + i\nu) - p_1^{n-1}(\mu_n^*(\nu;\beta) + i\nu)\big) = 0\,.$$

$\mu_n^*(0;\beta)$ is by definition the point of phase coexistence for the constrained pressures of order $n-1$, when $z = \mu \in \mathbb{R}$. The point of phase coexistence of the model is given by $\mu^*(0;\beta) = \lim_n \mu_n^*(0;\beta)$.

Proposition 3.1. *Let $0 < \varepsilon < \rho$, and set*

$$U_0 := (-C_1^{-1}\varepsilon, C_1^{-1}\varepsilon) \quad \textit{and} \quad \mathbb{U}_0 := \{z \in \mathbb{C} :\ \mathrm{Re} z \in U_0\}\,.$$

Then there exist (a) *$\delta = \delta(\beta)$ such that $\lim_{\beta\to\infty}\delta(\beta) = 0$, and* (b) *$\beta_0 \in \mathbb{R}^+$ such that if $\beta \geq \beta_0$, then*

$$\tau(\beta) := \beta(\rho - \varepsilon) - 3C_0\delta > O\,,$$

and the following holds.

(1) *There exists a continuous real-valued function on $\mathbb{R}$, $\nu \mapsto \mu^*(\nu;\beta) \in U_0$, so that $\mu^*(\nu;\beta) + i\nu \in \mathbb{U}_0$.*
(2) *If $\mu + i\nu \in \mathbb{U}_0$ and $\mu \leq \mu^*(\nu;\beta)$, then the weight $\omega(\Gamma^2)$ is $\tau(\beta)$-stable for all contours Γ^2 with boundary condition ψ_2, and analytic in $z = \mu + i\nu$ if $\mu < \mu^*(\nu;\beta)$.*
(3) *If $\mu + i\nu \in \mathbb{U}_0$ and $\mu \geq \mu^*(\nu;\beta)$, then the weight $\omega(\Gamma^1)$ is $\tau(\beta)$-stable for all contours Γ^1 with boundary condition ψ_1, and analytic in $z = \mu + i\nu$ if $\mu > \mu^*(\nu;\beta)$.*

Corollary 3.1. *At high β, the pressure of the model can be constructed as a real-analytic function $p(\mu,\beta) = g_2(\mu,\beta) - \mu h(\psi_2)$ on $\{\mu :\ \mu < \mu^*(0;\beta)\} \cap U_0$. This function has a complex analytic extension in $\{z = \mu + i\nu :\ \mu < \mu^*(\nu;\beta)\} \cap \mathbb{U}_0$, which is given by $g_2(z,\beta) - zh(\psi_2)$. Similarly, the pressure can be constructed as a real-analytic function $p(\mu,\beta) = g_1(\mu,\beta) - \mu h(\psi_1)$ on $\{\mu :\ \mu > \mu^*(0;\beta)\} \cap U_0$. This function has a complex analytic extension in $\{z = \mu + i\nu :\ \mu > \mu^*(\nu;\beta)\} \cap \mathbb{U}_0$, which is given by $g_1(z,\beta) - zh(\psi_1)$.*

I outline the structure of the proof of proposition 3.1, and prove only the weaker result, that there exists a continuous real-valued function on $\mathbb{R}$, $\nu \mapsto \mu^*(\nu;\beta) \in U_0$, so that $\mu^*(\nu;\beta) + i\nu \in \mathbb{U}_0$, and that at $\mu^*(\nu;\beta) + i\nu$ all contours are $\tau_\star(\beta)$-stable, with

$$\tau_\star(\beta) := \beta\big(\rho(1-\theta^\star) - \varepsilon\big) \quad \text{for some } 0 < \theta^\star < 1. \tag{3.11}$$

This gives a constructive definition of the point of phase coexistence $\mu^*(\beta)$. The existence of two different phases follows from a standard Peierls argument[36], since

[36] The Peierls argument is at the origin of many works in statistical mechanics, and the notion of contour models has its origin in [Pe], which is worth while to read. It is interesting to notice that this paper became well-known only in the late sixties. Peierls himself wrote in [Pe] in 1936: *The Ising model is therefore now only of mathematical interest. Since, however, the problem of Ising's model in more than one dimension has led to a good deal of controversy and in particular since the opinion has often been expressed that the solution of the three-dimensional problem could be reduced to that of the linear model and would lead to similar results, it may be worth while to give its solution.* What Peierls did, was to show that the law of large numbers does not hold at

all contours are $\tau_\star(\beta)$-stable at $\mu^*(\beta)$. This proves the first statement of theorem 2.1.

Before giving the proof, I put into evidence a key step concerning the stability of the weights of contours. I introduce an auxiliary parameter $0 < \theta' < 1$, so that

$$\rho(1-\theta') > \varepsilon\,.$$

The parameter θ' enters into the size of the intervals U_n, see (3.17); the size of U_n is proportional to θ'. This parameter controls the volume term of the weight of a contour by the surface energy $\|\Gamma^q\|$ (see (3.15) and (3.16)). By taking ε smaller, one can choose θ' larger. $\theta^\star$ is chosen so that $\theta^\star > \theta'$ and $\rho(1-\theta^\star) > \varepsilon$. Set

$$\delta := K\mathrm{e}^{-\tau_\star(\beta)} \quad (K \text{ the constant in lemma 3.1}). \tag{3.12}$$

Let β_0 be large enough, and assume that $\beta \geq \beta_0$, and that for $q = 1, 2$, the weights $\omega_{n-1}(\Gamma^q)$ are $\tau_\star(\beta)$-stable and

$$\Big|\frac{d}{dz}\omega_{n-1}(\Gamma^q)\Big| \leq \mathrm{e}^{-\tau_\star(\beta)|\Gamma^q|}\,.$$

From (3.8) and lemma 3.1 one obtains

$$\Big|\frac{d}{dz}\big(p_2^{n-1} - p_1^{n-1}\big) + \Delta\Big| = \Big|\frac{d}{dz}\big(g_2^{n-1} - g_1^{n-1}\big)\Big| \leq 2\delta\,, \tag{3.13}$$

and ($m \neq q$)

$$\begin{aligned}
\big|\ln \Theta_q^{n-1}(\mathrm{Int}_m\,\Gamma^q) - \beta g_q^{n-1}\,V(\Gamma^q)|\big| &\leq \delta\, C_0|\Gamma^q| \\
\big|\ln \Theta_m^{n-1}(\mathrm{Int}_m\,\Gamma^q) - \beta g_m^{n-1}\,V(\Gamma^q)|\big| &\leq \delta\, C_0|\Gamma^q|\,.
\end{aligned}$$

Let Γ^q be a contour with $V(\Gamma^q) = n$. Then (always $m \neq q$)

$$\begin{aligned}
|\omega(\Gamma^q)| &= \exp\big[-\beta\mathrm{Re}\mathcal{H}(\varphi_{\Gamma^q}|\psi_q)\big]\,\Big|\frac{\Theta_m(\mathrm{Int}_m\,\Gamma^q)}{\Theta_q(\mathrm{Int}_m\,\Gamma^q)}\Big| \\
&\leq \exp\Big[-\beta\|\Gamma^q\| + \big(\beta\varepsilon + 2C_0\delta\big)|\Gamma^q| + \beta\mathrm{Re}\big(p_m^{n-1} - p_q^{n-1}\big)V(\Gamma^q)\Big]\,,
\end{aligned} \tag{3.14}$$

because all contours inside $\mathrm{Int}_m\,\Gamma^q$ have a volume smaller than $n-1$, and (see (3.5))

$$|\mathrm{Re} z\, a(\varphi_{\Gamma^q})| \leq \varepsilon \quad \forall\, z \in \mathbb{U}_0\,.$$

To prove the stability of $\omega(\Gamma^q)$ one must control the volume term in the right-hand side of inequality (3.14). If

$$\mathrm{Re}\big(p_1^{n-1} - p_2^{n-1}\big)V(\Gamma^2) \leq \theta'\|\Gamma^2\| \tag{3.15}$$

and

$$\mathrm{Re}\big(p_2^{n-1} - p_1^{n-1}\big)V(\Gamma^1) \leq \theta'\|\Gamma^1\|\,, \tag{3.16}$$

then $\omega(\Gamma^2)$ and $\omega(\Gamma^1)$ are $\tau_\star(\beta)$-stable. Indeed, these inequalities imply

$$\begin{aligned}
|\omega(\Gamma^q)| &\leq \exp\Big[-\beta(1-\theta')\|\Gamma^q\| + \big(\beta\varepsilon + 2C_0\delta\big)|\Gamma^q|\Big] \\
&\leq \exp\Big[-\beta\big((1-\theta^\star)\rho - \varepsilon\big)|\Gamma^q|\Big]\,.
\end{aligned}$$

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low temperature. For a finite Ising model with free boundary condition he proved that at least three quarters of the spins have the same value if the temperature is low enough. Free boundary condition, on the other hand, implies that the mean magnetization is zero (by symmetry). The failure of the law of large numbers is the consequence of the coexistence of two distinct phases, with two distinct non-zero values of the magnetization. The Peierls argument is also very nicely exposed in [Gri].

Verification of the inequalities (3.15) and (3.16) is possible because (3.13) provides a sharp estimate of the derivative of $p_2^{n-1} - p_1^{n-1}$. Details are given below.

Proof. Let θ' be chosen as above, and $b_0 := \varepsilon C_1^{-1}$. $p_q^0(\mu + i\nu)$ is defined on the interval $U_0(\nu;\beta) := (-b_0, b_0)$, and set $\mu_0^*(\nu;\beta) := 0$. The two decreasing sequences $\{b_n^q\}$, $q = 1,2$ and $n \geq 1$, are chosen as

$$b_n^1 \equiv b_n^2 := \frac{\chi\theta'}{(\Delta + 2\delta)n^{\frac{1}{d}}}\,,\ n \geq 1\,. \tag{3.17}$$

The constant χ is the best constant such that

$$V(\Gamma^q)^{\frac{d-1}{d}} \leq \chi^{-1}\|\Gamma^q\| \quad \forall\,\Gamma^q\,,\, q = 1,2\,. \tag{3.18}$$

It is immediate to verify, when β is large enough, that

$$b_n^q - b_{n+1}^q > \frac{2\delta^{l(n)}}{\beta(\Delta - 2\delta)}\,, \quad \forall n \geq 1\,. \tag{3.19}$$

On $\mathbb{U}_0$ all contours Γ with volume zero are $\beta(\rho-\varepsilon)$-stable, and, if β_0 is large enough,

$$\Big|\frac{d}{dz}\omega(\Gamma)\Big| \leq \beta C_1|\Gamma|\mathrm{e}^{-\beta(\rho-\varepsilon)|\Gamma|} \leq \beta C_1 \mathrm{e}^{-[\beta(\rho-\varepsilon)-1]|\Gamma|} \leq \mathrm{e}^{-\tau_\star(\beta)|\Gamma|}\,.$$

The proof of proposition 3.1 consists of proving iteratively the following four statements.

A. There exists a unique continuous solution $\nu \mapsto \mu_n^*(\nu;\beta)$ of the equation
$$\mathrm{Re}\big(p_2^{n-1}(\mu_n^*(\nu;\beta) + i\nu) - p_1^{n-1}(\mu_n^*(\nu;\beta) + i\nu)\big) = 0\,,$$
so that (3.10) holds.

B. For any contour Γ^q $\omega_n(\Gamma^q)$ is well-defined and analytic on $\mathbb{U}_n$, and $\omega_n(\Gamma^q)$ is $\tau_\star(\beta)$-stable. Moreover, $\Theta_q^n(\Lambda) \neq 0$ for any finite Λ, and $p_q^n(z;\beta)$ is analytic on $\mathbb{U}_n$.

C. On $\mathbb{U}_n$, $\big|\frac{d}{dz}\omega_n(\Gamma^q)\big| \leq \mathrm{e}^{-\tau_\star(\beta)|\Gamma^q|}$.

D. If $z = \mu + i\nu \in \mathbb{U}_0$ and $\mu \leq \mu_n^*(\nu;\beta) - b_n^1$, then $\omega(\Gamma^2)$ is $\tau(\beta)$-stable for any Γ^2 with boundary condition ψ_2. If $z = \mu + i\nu \in \mathbb{U}_0$ and $\mu \geq \mu_n^*(\nu;\beta) + b_n^2$, then $\omega(\Gamma^1)$ is $\tau(\beta)$-stable for any Γ^1 with boundary condition ψ_1.

The main technical part is the proof of point D. The argument for proving D is due to [Z] (see [FrPf1]). I prove only A, B and C.

Assume that the construction has been done for all $k \leq n-1$.

A. Proof of the existence of $\mu_n^*(\nu;\beta) \in \mathbb{U}_{n-1}$.
$\mu_n^*(\nu;\beta)$ is solution of the equation

$$\mathrm{Re}\big(p_2^{n-1}(\mu_n^*(\nu;\beta) + i\nu) - p_1^{n-1}(\mu_n^*(\nu;\beta) + i\nu)\big) = 0\,.$$

The value of ν is fixed, and set

$$F^k(\mu) := p_2^k(\mu + i\nu) - p_1^k(\mu + i\nu)\,.$$

One proves that

$$\mu \mapsto \mathrm{Re}F^{n-1}(\mu)$$

is strictly decreasing, and takes positive and negative values. If $\mu'+i\nu\in\mathbb{U}_{n-1}$, then

$$\begin{aligned}F^{n-1}(\mu') &= F^{n-1}(\mu')-F^{n-2}(\mu^*_{n-1}) \\ &= F^{n-1}(\mu')-F^{n-1}(\mu^*_{n-1})+F^{n-1}(\mu^*_{n-1})-F^{n-2}(\mu^*_{n-1}) \\ &= \int_{\mu^*_{n-1}}^{\mu'}\frac{d}{d\mu}F^{n-1}(\mu)\,d\mu+\big(g_2^{n-1}-g_2^{n-2}\big)(\mu^*_{n-1}+i\nu) \\ &\quad -\big(g_1^{n-1}-g_1^{n-2}\big)(\mu^*_{n-1}+i\nu)\,.\end{aligned}\tag{3.20}$$

If $V(\Gamma)=n-1$, then $|\Gamma|\geq l(n-1)$. Therefore, by lemma 3.1,

$$|\big(g_q^{n-1}-g_q^{n-2}\big)(\mu^*_{n-1}+i\nu)|\leq\beta^{-1}\delta^{l(n-1)}\,.\tag{3.21}$$

If $z'=\mu'+i\nu\in\mathbb{U}_{n-1}$, then (3.20), (3.13) and (3.21) imply

$$\begin{aligned}-\Delta(\mu'-\mu^*_{n-1})-2\delta|\mu'-\mu^*_{n-1}|-2\beta^{-1}\delta^{l(n-1)} &\leq \mathrm{Re}F^{n-1}(z') \\ &\leq -\Delta(\mu'-\mu^*_{n-1})+2\delta|\mu'-\mu^*_{n-1}|+2\beta^{-1}\delta^{l(n-1)}\,.\end{aligned}$$

Since (3.19) holds,

$$b^q_{n-1}>b^q_{n-1}-b^q_n>\frac{2\delta^{l(n-1)}}{\beta(\Delta-2\delta)}\,,$$

so that $\mathrm{Re}F^{n-1}(\mu^*_{n-1}-b^1_{n-1})>0$ and $\mathrm{Re}F^{n-1}(\mu^*_{n-1}+b^2_{n-1})<0$. This proves the existence of μ^*_n and its uniqueness, since $\mu\mapsto\mathrm{Re}F^{n-1}(\mu)$ is strictly decreasing (see (3.13)). Moreover, choosing $\mu'=\mu^*_n(\nu;\beta)$ in (3.20), one gets

$$|\mu^*_n(\nu;\beta)-\mu^*_{n-1}(\nu;\beta)|\leq\frac{2\delta^{l(n-1)}}{\beta(\Delta-2\delta)}\,.$$

Therefore $\mathbb{U}_n\subset\mathbb{U}_{n-1}$. The implicit function theorem implies that $\nu\mapsto\mu^*_n(\nu;\beta)$ is continuous (even C^∞).

B. Proof of the $\tau_\star$-stability on $\mathbb{U}_n$ of the weights $\omega_n(\Gamma^q)$ of all contours Γ^q, $q=1,2$. By the induction hypothesis the weights $\omega_n(\Gamma^q)$ are analytic in $\mathbb{U}_{n-1}$. This implies that p^n_q is analytic on $\mathbb{U}_n$. The proof of the stability has been already outlined. Let Γ^q be a contour with $V(\Gamma^q)=n$. One verifies (3.15), if $\mu\leq\mu^*_n+b^2_n$, and (3.16), if $\mu\geq\mu^*_n-b^1_n$. The choice of $\{b^q_n\}$ and the isoperimetric inequality (3.18) imply

$$\begin{aligned}\big|\mathrm{Re}\big(p_m^{n-1}-p_q^{n-1}\big)\big|\frac{V(\Gamma^q)}{\|\Gamma^q\|} &= \Big|\mathrm{Re}\int_{\mu^*_n}^{\mu}\frac{d}{d\mu}\big(p_m^{n-1}-p_q^{n-1}\big)\,d\mu\Big|\frac{V(\Gamma^q)}{\|\Gamma^q\|} \\ &\leq |\mu-\mu^*_n|(\Delta+2\delta)\frac{V(\Gamma^q)}{\|\Gamma^q\|} \\ &\leq b_n(\Delta+2\delta)V(\Gamma^q)^{\frac{1}{d}}\chi^{-1} \\ &\leq \theta'\,.\end{aligned}$$

C. Proof of the $\tau_\star$-stability of $\frac{d}{dz}\omega_n(\Gamma^q)$ on $\mathbb{U}_n$.
Let $V(\Gamma^q)=n$; from (3.4)

$$\begin{aligned}\frac{d}{dz}\omega_n(\Gamma^q) &= \omega_n(\Gamma^q)\Big(-\beta a(\varphi_{\Gamma^q})-\beta\big(h(\psi_m)-h(\psi_q)\big)V(\Gamma^q) \\ &\quad +\frac{d}{dz}\big(\ln\Theta_m(\mathrm{Int}_m\,\Gamma^q)-\ln\Theta_q(\mathrm{Int}_m\,\Gamma^q)\big)\Big)\,.\end{aligned}$$

There exists a constant[37] C_3 such that

$$\begin{aligned}\Big|\frac{d}{dz}\omega_n(\Gamma^q)\Big| &\leq \beta|\omega_n(\Gamma^q)|\big(|\Gamma^q|(C_1+2\delta C_0)+V(\Gamma^q)(\Delta+2\delta)\big)\\ &\leq \beta C_3|\omega_n(\Gamma^q)||\Gamma^q|^{\frac{d}{d-1}}\\ &\leq \mathrm{e}^{-\tau_\star(\beta)|\Gamma^q|}\,,\end{aligned}$$

provided that β_0 is large enough (use (3.14) for controlling $|\omega_n(\Gamma^q)|$). □

It is not difficult to prove more regularity for the curve $\nu\mapsto\mu^*(\nu;\beta)$. But this is not necessary for these lectures. If β is sufficiently large, then for all $n\geq 1$

$$\frac{d}{d\nu}\mu_n^*(0;\beta)=0\,,$$

and

$$\Big|\frac{d^2}{d\nu^2}\mu_n^*(\nu;\beta)\Big|\leq\frac{2\delta}{\Delta-2\delta}\Big(\Big(\frac{2\delta}{\Delta-2\delta}\Big)^2+\frac{2\delta}{\Delta-2\delta}+1\Big)\,. \tag{3.22}$$

The first formula is a consequence of the reality of the constrained pressures on the real axis, which implies that $\nu\mapsto\mu_n^*(\nu;\beta)$ is an even function. Moreover,

$$|\mu^*(\nu;\beta)-\mu_n^*(\nu;\beta)|\leq\frac{2\delta^{l(n)}}{\beta(\Delta-2\delta)}\,. \tag{3.23}$$

The next proposition gives an estimate for the derivative of the weight of a contour. It is a strengthening of point C above (see [FrPf1]).

Proposition 3.2. *Under the conditions of Proposition 3.1, there exist $\beta_0\in\mathbb{R}^+$ and a constant D so that the following holds for all $\beta\geq\beta_0$. Let*

$$\tau'(\beta):=\tau(\beta)-D\,.$$

(1) *If $\mu+i\nu\in\mathbb{U}_0$ and $\mu\leq\mu^*(\nu;\beta)$, then*

$$\Big|\frac{d}{dz}\omega(\Gamma^2)(z)\Big|\leq\beta C_3\mathrm{e}^{-\tau'(\beta)|\Gamma^2|}\,.$$

(2) *If $\mu+i\nu\in\mathbb{U}_0$ and $\mu\geq\mu^*(\nu;\beta)$, then*

$$\Big|\frac{d}{dz}\omega(\Gamma^1)(z)\Big|\leq\beta C_3\mathrm{e}^{-\tau'(\beta)|\Gamma^1|}\,.$$

[37] See (3.5), (3.18) and (3.6); for C_0, see footnote 35.

4. Proof of theorem 2.1

This section is devoted to the proof[38] of theorem 2.1 by the method due to Isakov [I1]. Whenever a specific boundary condition is needed, I choose the boundary ψ_2. The inverse temperature β is large, but fixed. There is a first order phase transition at

$$\mu^* := \mu^*(0,\beta)\,.$$

In the complex z-plane, there is a line of "transition points"[39], which is given by

$$\mathrm{Re} p_2(z^*) = \mathrm{Re} p_1(z^*)\,.$$

Since $z^* \equiv \mu^*(\nu;\beta) + i\nu$,

$$\mathrm{Re} g_1(z^*) - \mu^*(\nu;\beta) h(\psi_1) = \mathrm{Re} g_2(z^*) - \mu^*(\nu;\beta) h(\psi_2)\,.$$

With δ as in the proof of proposition 3.1, one has (see (3.23)), for any real ν,

$$|\mu^*(\nu;\beta)| \leq \frac{2\delta}{\beta\Delta}\,.$$

The first step, in subsection 4.1, is to obtain better complex neighbourhoods of μ^*, where the contours are stable and analytic functions of z. These neighbourhoods are of course contour-dependent; they depend only on the volume and the label of the contours. The size of these neighbourhoods is related directly to the best constant of variational problems of isoperimetric type. More specifically, for each $n \in \mathbb{N}$, there is a variational problem for each label q: find the best isoperimetric constant $\chi_q(n)$ such that

$$\chi_q(n)^{-1} := \inf\Big\{C : \frac{V(\Gamma^q)^{\frac{d-1}{d}}}{\|\Gamma^q\|} \leq C\,,\ \forall\, \Gamma^q \text{ such that } V(\Gamma^q) \geq n\Big\}\,. \tag{4.1}$$

In particular one has

$$V(\Gamma^q)^{\frac{d-1}{d}} \leq \chi_q\big(V(\Gamma^q)\big)^{-1} \|\Gamma^q\|\,.$$

The constants $\chi_q(n)$ form a bounded increasing sequence, and

$$\chi_q(\infty) := \lim_n \chi_q(n)\,.$$

These variational problems are *not those considered by Isakov*, and this difference is important.

Although the statement of theorem 2.1 makes sense only in the thermodynamic limit, most of its proof is done by considering finite-volume partition functions, in order to exploit the analyticity of the weights of contours in the neighbourhood of the transition point μ^*. Only at the very end of the proof one takes the thermodynamic limit. This last step is easy. Isakov's representation of the finite-volume partition function is defined in the next paragraph, and it is used in subsection 4.2 to express the k$^{\text{th}}$-derivative of the pressure in a convenient form.

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[38] I insist on the main points, omitting some details, which can be found in [FrPf1], in order to put into evidence the structure of the proof of theorem 2.1. It is a direct proof: estimate all derivatives, and show that the radius of convergence of the Taylor series at μ^* is zero! However, this is not simple, and the implementation of that program is long.

[39] This line is defined by the property that *all* contours are stable.

The partition function $\Theta_q(\Lambda)$ is expressed as a *finite* product of objects, indexed by the contours in Λ, so that

$$g_\Lambda^q := \frac{1}{\beta|\Lambda|} \ln \Theta_q(\Lambda) = -\frac{1}{\beta|\Lambda|} \sum_{\Gamma^q \subset \Lambda} u_\Lambda(\Gamma^q)\,, \tag{4.2}$$

is a sum with finitely many terms. Let $\Lambda = \Lambda(L)$ be the cubic box

$$\Lambda(L) := \{x \in \mathbb{Z}^d : |x| \leq L\}\,.$$

One introduces a linear order, denoted by $\leq$, among the finite set of all contours $\Gamma^q \subset \Lambda$ with boundary condition ψ_q. The linear order is such that $V(\Gamma'^q) \leq V(\Gamma^q)$ if $\Gamma'^q \leq \Gamma^q$. One chooses an enumeration of the contours, $\Gamma_1^q, \Gamma_2^q, \ldots$, so that the predecessor of Γ^q in that enumeration, denoted by $i(\Gamma^q)$, verifies $i(\Gamma^q) \leq \Gamma^q$ (if Γ^q is not the smallest contour). Then, one introduces restricted partition functions $\Theta_{\Gamma^q}(\Lambda)$,

$$\Theta_{\Gamma^q}(\Lambda) := 1 + \sum \prod_{i=1}^n \omega(\Gamma_i^{q\prime})\,, \tag{4.3}$$

where the sum is over all families of compatible contours $\{\Gamma_1^{q\prime}, \ldots, \Gamma_n^{q\prime}\}$ with the property that $\Gamma_j^{q\prime} \leq \Gamma^q$ for all j. The partition function $\Theta_q(\Lambda)$ is written as

$$\Theta_q(\Lambda) = \prod_{\Gamma^q \subset \Lambda} \frac{\Theta_{\Gamma^q}(\Lambda)}{\Theta_{i(\Gamma^q)}(\Lambda)}\,.$$

By convention $\Theta_{i(\Gamma^q)}(\Lambda) := 1$ when Γ^q is the smallest contour. Let

$$u_\Lambda(\Gamma^q) := -\ln \frac{\Theta_{\Gamma^q}(\Lambda)}{\Theta_{i(\Gamma^q)}(\Lambda)}\,. \tag{4.4}$$

$u_\Lambda(\Gamma^q)$ is the free energy cost for introducing the new contour Γ^q in the restricted model, where all contours satisfy $\Gamma'^q \leq i(\Gamma^q)$.

Subsection 4.3 is in some sense the core of the proof. The crucial observation of Isakov is that one gets accurate estimates of (large) derivatives of $u_\Lambda(\Gamma^q)$ by the stationary phase method, provided that $V(\Gamma^q)$ is large: there exists $k_0 \in \mathbb{N}$, and for each contour Γ^q an integer $k_+^q(\Gamma^q) \geq k_0$ (see (4.24)), such that for each $k \in [k_0, k_+^q(\Gamma^q)]$ the k$^{\text{th}}$-derivative of $u_\Lambda(\Gamma^q)$, at μ^*, can be *estimated uniformly in* Λ by the stationary phase method. The function $k_+^q(\Gamma^q)$ has the properties

$$k_+^q(\Gamma_1^q) < k_+^q(\Gamma_2^q)\,,\ \text{if}\ V(\Gamma_1^q) < V(\Gamma_2^q) \quad \text{and} \quad \lim_{V(\Gamma^q)\to\infty} k_+^q(\Gamma^q) = \infty\,.$$

Let $k_0 < k \in \mathbb{N}$ be given. Using the function $k_+^q(\Gamma^q)$, one distinguishes

$$k\textsf{-large contours},\ \text{if}\ k \leq k_+^q(\Gamma^q) \quad \text{and} \quad k\textsf{-small contours},\ \text{if}\ k > k_+^q(\Gamma^q).$$

The k$^{\text{th}}$-derivative of g_Λ^q at μ^* is

$$[g_\Lambda^q]_{\mu^*}^{(k)} = -\frac{1}{\beta|\Lambda|} \sum_{\Gamma^q \subset \Lambda} [u_\Lambda(\Gamma^q)]_{\mu^*}^{(k)}\,.$$

The contribution of $[u_\Lambda(\Gamma^q)]_{\mu^*}^{(k)}$ to the k$^{\text{th}}$-derivative of the pressure, when Γ^q is a large contour[40], is controlled uniformly in the box Λ. The contributions of $|[u_\Lambda(\Gamma^q)]_{\mu^*}^{(k)}|$,

[40] In fact, one must make a finer distinction between contours in the general case. One distinguish large and thin contours and large and fat contours. These finer details are treated in subsection 4.3. Only the contributions of large and thin contours are important.

for small contours, are estimated from above, in a straightforward manner, using the Cauchy formula. Let

$$s_\Lambda^{q,k} = \frac{1}{\beta|\Lambda|} \sum_{\substack{\Gamma^q \subset \Lambda \\ \Gamma^q \ k\text{-small}}} u_\Lambda(\Gamma^q)\,.$$

Then

$$\big|[s_\Lambda^{q,k}]_{\mu^*}^{(k)}\big| = \Big|\frac{k!}{2\pi i}\oint_{\partial D_r} \frac{s_\Lambda^{q,k}(z)}{(z-\mu^*)^{k+1}}\,dz\Big| \leq \frac{k!}{r^k} \sup_{z\in\partial D_r} |s_\Lambda^{q,k}(z)|\,,$$

where the disc D_r of center μ^* and radius r is taken as large as possible, according to the results of subsection 4.1.

The next step is to show that the contribution of large contours to $[g_\Lambda^q]_{\mu^*}^{(k)}$ dominates that of small contours. This delicate analysis, which works only at low temperature, is presented in subsection 4.4. Two basic facts are used:
(1) the sign of $[u_\Lambda(\Gamma^q)]_{\mu^*}^{(k)}$, for any large contour Γ^q, is the *same* (see (4.10));
(2) given $\varepsilon > 0$ there exists $\chi_q(\varepsilon)$ and $n(\varepsilon)$, such that

$$(1+\varepsilon)\chi_q(\varepsilon) > \chi_q(\infty) \quad \text{and} \quad \chi_q(\infty) \geq \chi_q(n) \geq \chi_q(\varepsilon) \ \text{ if } n \geq n(\varepsilon)\,. \tag{4.5}$$

By definition of the variational problems (4.1), there exists Γ_n^q, $n \geq n(\varepsilon)$, such that

$$\lim_{n\to\infty} \|\Gamma_n^q\| = \infty \quad \text{and} \quad V(\Gamma_n^q)^{\frac{d-1}{d}} \geq \frac{\|\Gamma_n^q\|}{(1+\varepsilon)\chi_q(\varepsilon)}\,. \tag{4.6}$$

Let

$$k_n := \Big\lfloor \frac{d-1}{d}\beta\|\Gamma_n^q\| \Big\rfloor\,.$$

One verifies that Γ_n^q is a k_n-large (and thin) contour, and that

$$[u_\Lambda(\Gamma_n^q))]_{\mu^*}^{(k_n)} \sim B^{k_n}\,(k_n!)^{\frac{d}{d-1}} \quad (\text{for large enough } \Lambda)\,.$$

Moreover, the contribution of $[u_\Lambda(\Gamma_n^q))]_{\mu^*}^{(k_n)}$ to the $\mathrm{k_n}^{\mathrm{th}}$-derivative of g_Λ^q at μ^* dominates that of $[s_\Lambda^{q,k_n}]_{\mu^*}^{(k_n)}$. Hence, for some constant C,

$$[g_\Lambda^q]_{\mu^*}^{(k_n)} \sim C^{k_n}\,(k_n!)^{\frac{d}{d-1}} \quad (\text{for large enough } \Lambda)\,.$$

Since this bound is uniform in Λ, and (see lemma 4.3 in [FrPf1])

$$\lim_{L\to\infty} [g^2_{\Lambda(L)}]_{\mu^*}^{(k)} = \lim_{t\uparrow\mu^*} [g^2]_t^{(k)} \quad \text{or} \quad \lim_{L\to\infty} [g^1_{\Lambda(L)}]_{\mu^*}^{(k)} = \lim_{t\downarrow\mu^*} [g^1]_t^{(k)}\,,$$

the same result holds in the thermodynamic limit. This proves theorem 2.1.

To summarize, the main steps of the proofs are:

(1) Write the partition function as in (4.2);

(2) Make the best possible (essentially optimal) analytic extension for each term $u_\Lambda(\Gamma^q)$ of this sum.

(3) Estimate $[u_\Lambda(\Gamma^q))]_{\mu^*}^{(k)}$, for large contours Γ^q, by the stationary phase method applied to a Cauchy integral suitably chosen. (The Cauchy integral is independent of the choice of ∂D_r, but the stationary phase method applies only at a specific value $r(k)$ of r.)

(4) Use the existence of the contours Γ_n^q, with $\lim_n V(\Gamma_n^q)=\infty$, which almost solve the variational problems (4.1), in order show the existence of a constant C so that

$$[g_\Lambda^q]_{\mu^*}^{(k_n)} \sim C^{k_n}\,(k_n!)^{\frac{d}{d-1}} \quad \text{with} \quad k_n := \left\lfloor \frac{d-1}{d}\beta\|\Gamma_n^q\| \right\rfloor .$$

Remark 4.1. Before giving some details of the implementation of this program I show heuristically why it may work. For this, I assume that the isoperimetric inequality is

$$\chi_2^* V(\Gamma^2)^{\frac{d-1}{d}} \leq \|\Gamma^2\| , \tag{4.7}$$

and that there are contours which saturate (4.7), with arbitrary large volumes. For large contours (see (4.13), (4.14), and (3.13), (3.14)),

$$-u_\Lambda(\Gamma^2) \approx \phi_\Lambda(\Gamma^2) \approx \omega(\Gamma^2) \approx \mathrm{e}^{-\beta\|\Gamma^2\|+(z-\mu^*)\beta\Delta V(\Gamma^2)} . \tag{4.8}$$

Therefore, one expects to have an analytical continuation of $u_\Lambda(\Gamma^2)$ in a complex neighbourhood of μ^*, which is

$$\left\{ z : |z-\mu^*| < \Delta^{-1}\frac{\chi_2^*}{V(\Gamma^2)^{\frac{1}{d}}} \right\} ,$$

because for large β one expects that a contour whose volume is smaller than $V(\Gamma^2)$ is stable in this neighbourhood.

As already mentioned, the core of the proof is to have sharp estimates of the k^{th}-derivative of $u_\Lambda(\Gamma^2)$ for large contours. One can show[41] that if $k \leq k_+^2(\Gamma^2)$, with

$$k_+^2(\Gamma^2) \sim \beta\chi_2^* V(\Gamma^2)^{\frac{d-1}{d}} , \tag{4.9}$$

then not only $u_\Lambda(\Gamma^2)$ is approximately given by (4.8), but that

$$[u_\Lambda(\Gamma^2)]_{\mu^*}^{(k)} \sim (\Delta\beta V(\Gamma^2))^k \mathrm{e}^{-\beta\|\Gamma^2\|} . \tag{4.10}$$

If Γ^2 saturates inequality (4.7), $\chi_2^* V(\Gamma^2)^{\frac{d-1}{d}} = \|\Gamma^2\|$, then

$$(\Delta\beta V(\Gamma^2))^k \mathrm{e}^{-\beta\|\Gamma^2\|} = (\Delta\beta)^k \mathrm{e}^{-\beta\chi_2^* V(\Gamma^2)^{\frac{d-1}{d}} + k\ln V(\Gamma^2)} .$$

This quantity is maximal when

$$k = \frac{d-1}{d}\,\beta\chi_2^* V(\Gamma^2)^{\frac{d-1}{d}} < k_+^2(\Gamma^2) .$$

Therefore this contour, say Γ_*^2, is k-large. Since $-[u_\Lambda(\Gamma^2)]_{\mu^*}^{(k)} \geq 0$ for all k-large contours, one has

$$-\sum_{\Gamma^2 :\, k\text{-large}} [u_\Lambda(\Gamma^2)]_{\mu^*}^{(k)} \geq [u_\Lambda(\Gamma_*^2)]_{\mu^*}^{(k)} ,$$

and

$$\begin{aligned} -[u_\Lambda(\Gamma_*^2)]_{\mu^*}^{(k)} &\sim (\Delta\beta)^k \Big(\frac{k}{\beta\chi_2^*}\frac{d}{d-1}\Big)^{k\frac{d}{d-1}} \mathrm{e}^{-k\frac{d}{d-1}} \\ &\sim \Big[\Delta \frac{1}{\beta^{\frac{1}{d-1}}}\frac{1}{\chi_2^{*\frac{d}{d-1}}}\Big(\frac{d}{d-1}\Big)^{\frac{d}{d-1}}\Big]^k k!^{\frac{d}{d-1}} . \end{aligned}$$

On the other hand, a k-small contour has a volume at most equal to

$$V(\Gamma^2) \sim \Big(\frac{k}{\beta\chi_2^*}\Big)^{\frac{d}{d-1}} ,$$

[41] See remark 4.2 in subsection 4.3 for a simple argument for justifying (4.9).

which implies that one can choose a disc of radius

$$r \sim \Delta^{-1} \chi_2^{*\frac{d}{d-1}} \beta^{\frac{1}{d-1}} k^{-\frac{1}{d-1}}$$

in order to estimate the contribution of the k-small contours to $[g_\Lambda^2]_{\mu^*}^{(k)}$. This contribution is at most of the order

$$\Big(\Delta \frac{1}{\beta^{\frac{1}{d-1}} \chi_2^{*\frac{d}{d-1}}}\Big)^k k!\, k^{\frac{k}{d-1}} \approx \Big[\Delta \frac{1}{\beta^{\frac{1}{d-1}} \chi_2^{*\frac{d}{d-1}}} \mathrm{e}^{\frac{1}{d-1}}\Big]^k k!^{\frac{d}{d-1}} \,.$$

Since

$$\frac{d}{(d-1)} > \mathrm{e}^{\frac{1}{d}} \,,$$

it is dominated by that of the k-large contours. Indeed,

$$\begin{aligned} d\Big(\mathrm{e}^{\frac{1}{d}} - 1\Big) &= d\Big(\mathrm{e}^{\frac{1}{d}} - 1 - \frac{1}{d} + \frac{1}{d}\Big) = \sum_{n \geq 2} \frac{1}{n!}\Big(\frac{1}{d}\Big)^{n-1} + 1 \\ &= 1 + \sum_{n \geq 1} \frac{1}{(n+1)!}\Big(\frac{1}{d}\Big)^n \\ &< 1 - \frac{1}{2d} + \sum_{n \geq 1} \frac{1}{n!}\Big(\frac{1}{d}\Big)^n = \mathrm{e}^{\frac{1}{d}} - \frac{1}{2d} \,. \end{aligned}$$

4.1. **Analytic continuation and isoperimetric problems.** It is not very difficult to show that the weight $\omega(\Gamma^2)$ has an analytic continuation in a disc of radius $O(V(\Gamma^2)^{-\frac{1}{d}})$ centered at μ^*. This is not sufficient. Let[42]

$$R_q(n) := \inf_{m: m \leq n} \frac{\chi_q(m)}{m^{\frac{1}{d}}} \,,$$

where $\chi_q(n)$ is the isoperimetric constant in (4.1).

Lemma 4.1. *For any $\chi'_q < \chi_q(\infty)$, there exists $N(\chi'_q)$ such that for all $n \geq N(\chi'_q)$,*

$$\frac{\chi'_q}{n^{\frac{1}{d}}} \leq R_q(n) \leq \frac{\chi_q(\infty)}{n^{\frac{1}{d}}} \,.$$

For $q = 1, 2$, $n \mapsto nR_q(n)$ is increasing in n.

Proof. Let $q = 2$ and suppose that

$$R_2(n) = \frac{\chi_2(m)}{m^{\frac{1}{d}}} \quad \text{for } m < n.$$

Then $R_2(m') = R_2(n)$ for all $m \leq m' \leq n$. Let n' be the largest $n \geq m$ such that

$$R_2(n) = \frac{\chi_2(m)}{m^{\frac{1}{d}}} \,.$$

One has $n' < \infty$, otherwise

$$0 < R_2(m) = R_2(n) \leq \frac{\chi_2(\infty)}{n^{\frac{1}{d}}} \quad \forall\, n \geq m \,,$$

[42]One knows very little about the variational problems (4.1). Instead of making hypothesis about the behaviour of these problems, as Isakov did in [I2], the strategy is to avoid discussing them, as much as possible, despite of the fact that these problems enter in an essential manner in the proof. This is the reason for introducing $R_q(n)$. The property which one needs is that $n \mapsto nR_q(n)$ is increasing in n (see lemma 4.1).

which is impossible. Either

$$R_2(n') = \frac{\chi_2(n')}{n'^{\frac{1}{d}}} \quad \text{or} \quad R_2(n'+1) = \frac{\chi_2(n'+1)}{(n'+1)^{\frac{1}{d}}};$$

for all $k \geq n'+1$, since $\chi_2(m)$ is increasing,

$$R_2(k) = \inf_{m\leq k} \frac{\chi_2(m)}{m^{\frac{1}{d}}} = \inf_{n'\leq m\leq k} \frac{\chi_2(m)}{m^{\frac{1}{d}}} \geq \inf_{n'\leq m\leq k} \frac{\chi_2(n')}{m^{\frac{1}{d}}} = \frac{\chi_2(n')}{k^{\frac{1}{d}}}. \tag{4.11}$$

Inequality (4.11) is true for infinitely many n'; since there exists m such that $\chi_2' \leq \chi_2(m)$, the first statement is proved.

On an interval of constancy of $R_2(n)$, $n \mapsto n^a R_2(n)$ is increasing. On the other hand, if on $[m_1, m_2]$

$$R_2(n) = \frac{\chi_2(n)}{n^{\frac{1}{d}}},$$

then $n \mapsto nR_2(n)$ is increasing on $[m_1, m_2]$ since $n \mapsto \chi_2(n)$ and $n \mapsto n^{1-\frac{1}{d}}$ are increasing. □

The next proposition gives the domains of analyticity and the stability properties of the weights $\omega(\Gamma)$ needed for estimating the derivatives of the pressure.

Proposition 4.1. *Let $0 < \theta < 1$, $\theta < \theta^\star < 1$, and $0 < \varepsilon < 1$, so that*

$$\rho(1-\theta^\star) - \varepsilon > 0.$$

Then there exists $\beta_0' \geq \beta_0$, such that for all $\beta \geq \beta_0'$ $\omega(\Gamma^2)$ is analytic and $\tau_\star(\beta)$-stable in a complex neighborhood of

$$\big\{z \in \mathbb{C} : \mathrm{Re} z \leq \mu^*(\mathrm{Im} z;\beta) + \theta\Delta^{-1} R_2(V(\Gamma^2))\big\} \cap \mathbb{U}_0.$$

Moreover

$$\big|\frac{d}{dz}\omega(\Gamma^2)\big| \leq \mathrm{e}^{-\tau_\star(\beta)|\Gamma^2|}.$$

Similar properties hold for $\omega(\Gamma^1)$ in a complex neighborhood of

$$\big\{z \in \mathbb{C} : \mu^*(\mathrm{Im} z;\beta) - \theta\Delta^{-1} R_1(V(\Gamma^1)) \leq \mathrm{Re} z\big\} \cap \mathbb{U}_0.$$

$\tau_\star(\beta) = \beta(\rho(1-\theta^\star) - \varepsilon)$.

Proof. $\omega(\Gamma^2)$ is $\tau(\beta)$-stable if $\mathrm{Re} z \leq \mu^*(\nu;\beta) \cap \mathbb{U}_0$, and $\frac{d}{dz}\omega(\Gamma^2)$ is $\tau'(\beta)$-stable on the same region (propositions 3.1 and 3.2). Similar statements hold for $\omega(\Gamma^1)$ on $\mathrm{Re} z \geq \mu^*(\nu;\beta) \cap \mathbb{U}_0$. Let

$$I_n(\nu;\beta) := \big(\mu^*(\nu;\beta) - \theta\Delta^{-1}R_1(n), \mu^*(\nu;\beta) + \theta\Delta^{-1}R_2(n)\big). \tag{4.12}$$

As in the proof of B and C of proposition 3.1, one proves by iteration, that on the intervals $I_n(\nu;\beta)$, $\omega(\Gamma^q)$ and $\frac{d}{dz}\omega(\Gamma^q)$ are $\tau_\star(\beta)$-stable.

To prove the stability of $\omega(\Gamma^q)$ one verifies (3.15) and (3.16) for some $\theta' < \theta^\star$. Suppose that the statement is correct for $V(\Gamma^q) \leq n-1$. $\delta = \delta(\beta)$ is defined by

(3.12). Let $V(\Gamma^2)=n$, $z=\mu+i\nu$, and $\mu\geq\mu^*(\nu;\beta)$. Then

$$\begin{aligned}\mathrm{Re}\big(p_1^{n-1}(z)-p_2^{n-1}(z)\big)\frac{V(\Gamma^2)}{\|\Gamma^2\|}&=\mathrm{Re}\int_{\mu_n^*}^{\mu}\frac{d}{d\mu}\big(p_1^{n-1}(z)-p_2^{n-1}(z)\big)\frac{V(\Gamma^2)}{\|\Gamma^2\|}\\&\leq(\Delta+2\delta)\big(|\mu-\mu^*|+|\mu^*-\mu_n^*|\big)\frac{V(\Gamma^2)^{\frac{d-1}{d}}}{\|\Gamma^2\|}V(\Gamma^2)^{\frac{1}{d}}\\&\leq(\Delta+2\delta)\big(|\mu-\mu^*|+|\mu^*-\mu_n^*|\big)\frac{n^{\frac{1}{d}}}{\chi_2(n)}\\&\leq(\Delta+2\delta)\Big(|\mu-\mu^*|\frac{1}{R_2(n)}+|\mu^*-\mu_n^*|\frac{n^{\frac{1}{d}}}{\chi_2(n)}\Big)\\&\leq\frac{\Delta+2\delta}{\Delta}\theta+\frac{2(\Delta+2\delta)}{\beta(\Delta-2\delta)}\,\frac{\delta^{l(n)}n^{\frac{1}{d}}}{\chi_2(n)}\,.\end{aligned}$$

(3.23) is used for controlling $|\mu^*-\mu_n^*|$. If β is large enough, there exists $\theta<\theta'<\theta^\star$, independent of n, so that

$$\mathrm{Re}\big(p_1^{n-1}(z)-p_2^{n-1}(z)\big)\frac{V(\Gamma^2)}{\|\Gamma^2\|}\leq\theta'\,.$$

The stability of $\frac{d}{dz}\omega(\Gamma^2)$ is a consequence of (use (3.14) for controlling $|\omega_n(\Gamma^q)|$)

$$\Big|\frac{d}{dz}\omega(\Gamma^2)\Big|\leq\beta|\omega(\Gamma^2)|\big(|\Gamma^2|(C_1+2\delta C_0)+V(\Gamma^2)(\Delta+2\delta)\big)\leq\beta C_3|\Gamma^q|^{\frac{d}{d-1}}|\omega(\Gamma^2)|\,.$$

□

4.2. **Isakov's representation of the partition function.** The pressure g_Λ^q is defined in (4.2), and $u_\Lambda(\Gamma^q)$ in (4.4). Thus

$$[g_\Lambda^q]_{\mu^*}^{(k)}=-\frac{1}{\beta|\Lambda|}\sum_{\Gamma^q\subset\Lambda}[u_\Lambda(\Gamma^q)]_{\mu^*}^{(k)}\,.$$

One first writes $u_\Lambda(\Gamma^q)$ as follows.

$$\begin{aligned}\Theta_{\Gamma^q}(\Lambda)&=\Theta_{i(\Gamma^q)}(\Lambda)+\omega(\Gamma^q)\,\Theta_{i(\Gamma^q)}(\Lambda(\Gamma^q))\\&=\Theta_{i(\Gamma^q)}(\Lambda)\left(1+\omega(\Gamma^q)\,\frac{\Theta_{i(\Gamma^q)}(\Lambda(\Gamma^q))}{\Theta_{i(\Gamma^q)}(\Lambda)}\right)\,.\end{aligned}$$

In this expression $\Theta_{i(\Gamma^q)}(\Lambda(\Gamma^q))$ denotes the restricted partition function

$$\Theta_{i(\Gamma^q)}(\Lambda(\Gamma^q)):=1+\sum\prod_{i=1}^{n}\omega(\Gamma_i^q)\,,$$

where the sum is over all families of compatible contours $\{\Gamma_1^q,\ldots,\Gamma_n^q\}$, $\Gamma_i^q\leq i(\Gamma^q)$, $i=1,\ldots,n$, and such that $\{\Gamma^q,\Gamma_1^q,\ldots,\Gamma_n^q\}$ is a compatible family. Set

$$\phi_\Lambda(\Gamma^q):=\omega(\Gamma^q)\,\frac{\Theta_{i(\Gamma^q)}(\Lambda(\Gamma^q))}{\Theta_{i(\Gamma^q)}(\Lambda)}\,.\tag{4.13}$$

With these notations

$$u_\Lambda(\Gamma^q)=-\ln\big(1+\phi_\Lambda(\Gamma^q)\big)=\sum_{n\geq1}\frac{(-1)^n}{n}\phi_\Lambda(\Gamma^q)^n\,.\tag{4.14}$$

$[\phi_\Lambda(\Gamma^2)^n]^{(k)}_{\mu^*}$ is computed using the Cauchy formula,

$$[\phi_\Lambda(\Gamma^2)^n]^{(k)}_{\mu^*} = \frac{k!}{2\pi i}\oint_{\partial D_r} \frac{\phi_\Lambda(\Gamma^2)^n(z)}{(z-\mu^*)^{k+1}}\, dz\,,$$

where ∂D_r is the boundary of a disc D_r of radius r and center μ^* inside the analyticity region of proposition 4.1,

$$\mathbb{U}_0 \cap \big\{z\in\mathbb{C}:\, \mathrm{Re} z \leq \mu^*(\mathrm{Im}(z);\beta) + \theta\Delta^{-1}R_2(V(\Gamma^2))\big\}\,.$$

The function $z\mapsto \frac{\phi_\Lambda(\Gamma^2)^n(z)}{(z-\mu^*)^{k+1}}$ is real on the real axis, so that

$$\overline{\Big(\frac{\phi_\Lambda(\Gamma^2)^n(\overline{z})}{(\overline{z}-\mu^*)^{k+1}}\Big)} = \frac{\phi_\Lambda(\Gamma^2)^n(z)}{(z-\mu^*)^{k+1}}\,.$$

Consequently

$$\frac{k!}{2\pi i}\oint_{\partial D_r} \frac{\phi_\Lambda(\Gamma^2)^n(z)}{(z-\mu^*)^{k+1}}\, dz = \mathrm{Re}\Big\{\frac{k!}{2\pi i}\oint_{\partial D_r} \frac{\phi_\Lambda(\Gamma^2)^n(z)}{(z-\mu^*)^{k+1}}\, dz\Big\}\,. \tag{4.15}$$

Assuming[43] that the disc D_r is inside the analyticity region of $\omega(\Gamma^2)$, one decomposes ∂D_r into

$$\partial D_r^g := \partial D_r \cap \{z:\, \mathrm{Re} z \leq \mu^*(\mathrm{Im}(z);\beta) - \theta\Delta^{-1}R_1(V(\Gamma^2))\}$$

and

$$\partial D_r^d := \partial D_r \cap \{z:\, \mathrm{Re} z \geq \mu^*(\mathrm{Im}(z);\beta) - \theta\Delta^{-1}R_1(V(\Gamma^2))\}\,.$$

One writes (4.15) as a sum of two integrals $I^g_{k,n}(\Gamma^2)$ and $I^d_{k,n}(\Gamma^2)$,

$$I^g_{k,n}(\Gamma^2) := \mathrm{Re}\Big\{\frac{k!}{2\pi i}\oint_{\partial D_r^g} \frac{\phi_\Lambda(\Gamma^2)^n(z)}{(z-\mu^*)^{k+1}}\, dz\Big\} \tag{4.16}$$

and

$$I^d_{k,n}(\Gamma^2) := \mathrm{Re}\Big\{\frac{k!}{2\pi i}\oint_{\partial D_r^d} \frac{\phi_\Lambda(\Gamma^2)^n(z)}{(z-\mu^*)^{k+1}}\, dz\Big\}\,. \tag{4.17}$$

An analogous decomposition holds for Γ^1 instead of Γ^2.

4.3. **Estimate of $[u_\Lambda(\Gamma^q))]^{(k)}_{\mu^*}$ by the stationary phase method.** In order to apply the stationary phase method to evaluate $I^d_{k,n}(\Gamma^2)$, one makes a change of variable,

$$\zeta := z - \mu^*\,,$$

and writes $\phi_\Lambda(\Gamma^2)$ as

$$\phi_\Lambda(\Gamma^2)(\mu^*+\zeta) = \phi_\Lambda(\Gamma^2)(\mu^*)\,\mathrm{e}^{\beta\Delta V(\Gamma^2)(\zeta+\mathsf{g}(\Gamma^2)(\zeta))}\,, \tag{4.18}$$

where $\mathsf{g}(\Gamma^2)$ is an analytic function of ζ in a neighborhood of $\zeta=0$ and $\mathsf{g}(\Gamma^2)(0)=0$. Let

$$\mu^*\big(\mathrm{Im}(z);\beta\big) - \theta\Delta^{-1}R_1(V(\Gamma^2)) \leq \mathrm{Re} z \leq \mu^*\big(\mathrm{Im}(z);\beta\big) + \theta\Delta^{-1}R_2(V(\Gamma^2))\,.$$

---

[43]From (3.22) it follows that there exists C' independent of ν and n, such that

$$\mu^*_n(\nu;\beta) \geq \mu^*_n(0;\beta) - C'\nu^2\,.$$

This implies that the disc D_r of center μ^* and radius $r=\theta\Delta^{-1}R_2(V(\Gamma^2))$ is inside the analyticity region of $\omega(\Gamma^2)$, provided that $V(\Gamma^2)$ is large enough.

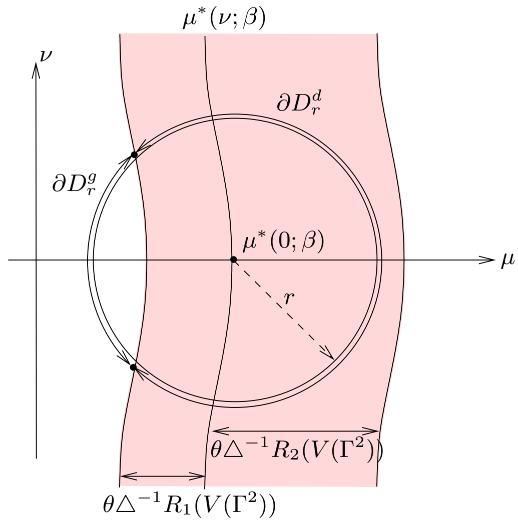

Figure 2: The decomposition of the integral into $I^g_{k,n}(\Gamma^2)$ and $I^d_{k,n}(\Gamma^2)$

In this region (see figure 2) one controls the weights of contours with boundary conditions ψ_2 and ψ_1. Therefore, one controls $\ln\Theta_1(\mathrm{Int}_1\,\Gamma^2)$, and

$$\phi_\Lambda(\Gamma^2)=\exp\Big[-\beta\mathcal{H}(\varphi_{\Gamma^2}|\psi_2)+\underbrace{\ln\frac{\Theta_1(\mathrm{Int}_1\,\Gamma^2)}{\Theta_2(\mathrm{Int}_1\,\Gamma^2)}+\ln\frac{\Theta_{i(\Gamma^2)}(\Lambda(\Gamma^2))}{\Theta_{i(\Gamma^2)}(\Lambda)}}_{:=\mathtt{G}(\Gamma^2)}\Big]\,.$$

By definition $z=\zeta+\mu^*$, so that (see (3.4))

$$\begin{aligned}-\beta\mathcal{H}(\varphi_{\Gamma^2}|\psi_2)(z)+\mathtt{G}(\Gamma^2)(z)&=-\beta\mathcal{H}(\varphi_{\Gamma^2}|\psi_2)(\mu^*)+\beta\Delta V(\Gamma^2)\zeta\\&\qquad-\beta a(\varphi_{\Gamma^2})\zeta+\int_{\mu^*}^{\mu^*+\zeta}\frac{d}{dz'}\mathtt{G}(\Gamma^2)(z')dz'+\mathtt{G}(\Gamma^2)(\mu^*)\\&=-\beta\mathcal{H}(\varphi_{\Gamma^2}|\psi_2)(\mu^*)+\mathtt{G}(\Gamma^2)(\mu^*)+\beta\Delta V(\Gamma^2)\zeta\\&\qquad+\underbrace{\int_{\mu^*}^{\mu^*+\zeta}\Big(\frac{d}{dz'}\mathtt{G}(\Gamma^2)(z')-\beta a(\varphi_{\Gamma^2})\Big)dz'}_{:=\beta\Delta V(\Gamma^2)\mathtt{g}(\Gamma^2)(\zeta)}\,.\end{aligned}$$

This proves (4.18).

$$\begin{aligned}\frac{d}{d\zeta}\mathtt{g}(\Gamma^2)(\zeta)=\frac{1}{\beta\Delta V(\Gamma^2)}\Big(&\frac{d}{d\zeta}\ln\Theta_1(\mathrm{Int}_1\,\Gamma^2)-\frac{d}{d\zeta}\ln\Theta_2(\mathrm{Int}_1\,\Gamma^2)\\&+\frac{d}{d\zeta}\ln\frac{\Theta_{i(\Gamma^2)}(\Lambda(\Gamma^2))}{\Theta_{i(\Gamma^2)}(\Lambda)}-\beta a(\varphi_{\Gamma^2})\Big)\,.\end{aligned}\tag{4.19}$$

The last term of the right-hand side of (4.19) is estimated using (3.5). The first two terms are estimated using proposition 4.1 and lemma 3.1. The third term is estimated by writing explicitly the logarithm of the quotient, using (3.9). After cancellation the resulting series is differentiated term by term and is estimated using the basic estimates of proposition 4.1 and lemma 3.1. For β large enough,

$$\Big|\frac{d}{d\zeta}\mathtt{g}(\Gamma^2)(\zeta)\Big|\leq C_7\,\mathrm{e}^{-\tau_\star(\beta)}+C_8\frac{|\Gamma^2|}{V(\Gamma^2)}\,,\tag{4.20}$$

for suitable constants C_7 and C_8. Moreover, there exists a constant C_9 so that

$$\exp\big[-\beta\|\Gamma^2\|(1+C_9\delta)\big] \leq \phi_\Lambda(\Gamma^2)(\mu^*) \leq \exp[-\beta\|\Gamma^2\|(1-C_9\delta)]\,. \tag{4.21}$$

Let

$$c(n) := n\beta\Delta V(\Gamma^2)\,.$$

The basic observation of Isakov is that some derivatives of $\phi_\Lambda(\Gamma^2)^n(\mu^*+\zeta)$ can be computed by the stationary phase method, and that the result is approximately the same as if $\phi_\Lambda(\Gamma^2)^n(\mu^*+\zeta)$ were an exponential function:

$$\phi_\Lambda(\Gamma^2)^n(\mu^*+\zeta) = \phi_\Lambda(\Gamma^2)^n(\mu^*)\,\mathrm{e}^{n\beta\Delta V(\Gamma^2)(\zeta+\cdots)} \approx \phi_\Lambda(\Gamma^2)^n(\mu^*)\,\mathrm{e}^{n\beta\Delta V(\Gamma^2)(\zeta)}\,.$$

Once this observation is done, then the mathematics is standard. In the next remark I illustrate how one can estimate the derivatives of e^{cz} at $z=0$.

Remark 4.2. I estimate the k^{th} derivative of e^{cz} at $z=0$. The Cauchy formula gives

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\mathrm{e}^{cz}\Big|_{z=0} = k!\,\frac{1}{2\pi i}\int_C \frac{\mathrm{e}^{cz}}{z^{k+1}}\,dz\,,$$

where I choose for C the boundary of a disc of radius r, centered at $z=0$. Let $z=r\mathrm{e}^{i\alpha}$. Then

$$\begin{aligned}\frac{1}{2\pi i}\int_C \frac{\mathrm{e}^{cz}}{z^{k+1}}\,dz &= \frac{1}{2\pi r^k}\int_{-\pi}^{\pi}\exp\big(cr\cos\alpha + i(cr\sin\alpha - k\alpha)\big)\,d\alpha \\ &= \frac{1}{2\pi r^k\mathrm{e}^{-cr}}\int_{-\pi}^{\pi}\exp\big(cr(\cos\alpha-1)\big)\,\cos\psi(\alpha)\,d\alpha\,,\end{aligned}$$

where $\psi(\alpha) := cr\sin\alpha - k\alpha$. (The term $i\sin\psi(\alpha)$ gives no contribution to the integral.) The radius r is a parameter, which is free, and one looks for α so that

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\big(\cos\alpha - 1\big) = 0 \quad\text{and}\quad \frac{\mathrm{d}}{\mathrm{d}\alpha}\psi(\alpha) = 0\,.$$

One gets two conditions

$$\sin\alpha = 0 \quad\text{and}\quad cr\cos\alpha - k = 0\,.$$

These two conditions can be satisfied with $\alpha=0$ and $cr=k$. Hence

$$\frac{1}{2\pi i}\int_C \frac{\mathrm{e}^{cz}}{z^{k+1}}\,dz = \frac{c^k}{2\pi k^k\mathrm{e}^{-k}}\int_{-\pi}^{\pi}\exp\big(k(\cos\alpha-1)\big)\,\cos\psi(\alpha)\,d\alpha\,.$$

One expands $\cos\alpha$ and $\psi(\alpha)$ around $\alpha=0$,

$$k(\cos\alpha-1) \simeq -\frac{k\alpha^2}{2} \quad\text{and}\quad \psi(\alpha) \simeq -\frac{k\alpha^3}{3!}\,.$$

I estimate here the k^{th} derivative of e^{-cx} at $x=0$ by replacing $\psi(\alpha)$ by 0 and by integrating over $\mathbb{R}$,

$$k!\,\frac{c^k}{2\pi k^k\mathrm{e}^{-k}}\int_{-\infty}^{\infty}\exp\Big(-\frac{kt^2}{2}\Big)\,dt = c^k\,\frac{k!}{\sqrt{2\pi k}k^k\mathrm{e}^{-k}} \simeq c^k\,,$$

using Stirling's formula for $k!$ in the denominator.

From this result it follows that one can estimate sharply the k^{th} derivative of e^{cz} at $z=0$ if one chooses in the Cauchy formula a disc with radius $r=k/c$. One can now justify (4.9). Applying this result to $\phi_\Lambda(\Gamma^2)$, with $\mathsf{g}(\Gamma^2)(\zeta)=0$ (see (4.18)), one expects to have sharp estimates for the k^{th} derivative if

$$\frac{k}{\beta\Delta V(\Gamma^2)}\leq \Delta^{-1}\chi_2^* V(\Gamma^2)^{\frac{1}{d}}\quad\text{(maximum radius which can be used)}\,,$$

i.e. if

$$k\leq k_+(\Gamma^2):=\beta V(\Gamma^2)^{\frac{d-1}{d}}\chi_2^*\,.$$

One parametrizes ∂D_r^d by $z:=\mu^*+r\mathrm{e}^{i\alpha}$, $-\alpha_1\leq\alpha\leq\alpha_2$, $0<\alpha_i\leq\pi$. Using (4.18),

$$I_{k,n}^d(\Gamma^2)=k!\frac{\phi_\Lambda^*(\Gamma^2)^n}{2\pi r^k}\int_{-\alpha_1}^{\alpha_2}\mathrm{e}^{c(n)r\cos\alpha+c(n)\mathrm{Re}\,\mathsf{g}(\Gamma^2)(\zeta)}\left[\cos(\widetilde{\psi}(\alpha))\right]d\alpha\,, \tag{4.22}$$

with

$$\widetilde{\psi}(\alpha):=c(n)r\sin\alpha+c(n)\,\mathrm{Im}\,\mathsf{g}(\Gamma^2)(\zeta)-k\alpha\,.$$

One searches for a stationary phase point $\zeta_{k,n}=r_{k,n}\mathrm{e}^{i\alpha_{k,n}}$ defined by the equations

$$\frac{d}{d\alpha}\Big(c(n)r\cos\alpha+c(n)\mathrm{Re}\,\mathsf{g}(\Gamma^2)\big(r\mathrm{e}^{i\alpha}\big)\Big)=0\quad\text{and}\quad\frac{d}{d\alpha}\widetilde{\psi}(\alpha)=0\,.$$

These equations are equivalent to the equations ($'$ denotes the derivative with respect to ζ)

$$c(n)\sin\alpha\big(1+\mathrm{Re}\,\mathsf{g}(\Gamma^2)'(\zeta)\big)+\cos\alpha\mathrm{Im}\,\mathsf{g}(\Gamma^2)'(\zeta)=0\,;$$
$$c(n)r\cos\alpha\big(1+\mathrm{Re}\,\mathsf{g}(\Gamma^2)'(\zeta)\big)-r\sin\alpha\mathrm{Im}\,\mathsf{g}(\Gamma^2)'(\zeta)=k\,.$$

Since $\mathsf{g}(\Gamma^2)$ is real on the real axis, $\alpha_{k,n}=0$ and $r_{k,n}$ is solution of

$$c(n)r\big(1+\mathsf{g}(\Gamma^2)'(r)\big)=k\,. \tag{4.23}$$

Lemma 4.2. *Let $\alpha_i\geq\pi/4$, $i=1,2$, $A\leq 1/25$ and $c(n)\geq 1$. If $\mathsf{g}(\zeta)$ is analytic in ζ in the disc $\{\zeta:\,|\zeta|\leq\kappa\}$, real on the real axis, and for all ζ in that disc*

$$\Big|\frac{d}{d\zeta}\mathsf{g}(\Gamma^2)(\zeta)\Big|\leq A\,,$$

then there exists $k_0(A)\in\mathbb{N}$, such that for all integers k,

$$k\in\big[k_0(A),c(n)(1-2\sqrt{A})\kappa\big]\,,$$

there is a unique solution $0<r_{k,n}<\kappa$ of (4.23). Moreover,

$$\frac{\mathrm{e}^{c(n)r_{k,n}+c(n)\,\mathsf{g}(\Gamma^2)(r_{k,n})}}{10\sqrt{c(n)r_{k,n}}}\leq\frac{1}{2\pi}\int_{-\alpha_1}^{\alpha_2}\mathrm{e}^{c(n)r\cos\alpha+c(n)\mathrm{Re}\,\mathsf{g}(\Gamma^2)}\left[\cos(\widetilde{\psi}(\alpha))\right]d\alpha$$
$$\leq\frac{\mathrm{e}^{c(n)r_{k,n}+c(n)\,\mathsf{g}(\Gamma^2)(r_{k,n})}}{\sqrt{c(n)r_{k,n}}}\,.$$

Proof. Existence and uniqueness of $r_{k,n}$ is a consequence of the monotonicity of $r\mapsto c(n)r\big(1+\mathsf{g}(\Gamma^2)'(r)\big)$. The estimates of the integral are obtained as above, by expanding around the stationary phase point. The details of that computation are given in appendix of [I1]. □

One defines

$$k_+(\Gamma^q|\theta,A,\beta):=\theta(1-2\sqrt{A})\beta V(\Gamma^q)R_q(V(\Gamma^q))\,. \tag{4.24}$$

Under the conditions of lemma 4.2, if

$$k \in \big[k_0(A), k_+(\Gamma^2|\theta, A, \beta)\big]\,,$$

then

$$k - \frac{kA}{(1+A)} = \frac{k}{(1+A)} \leq c(n) r_{k,n} \leq \frac{k}{(1-A)} = k + \frac{kA}{(1-A)}\,,$$

and

$$c(n)|\mathsf{g}(\Gamma^2)(r_{k,n})| = c(n)\,\Big|\int_0^{r_{k,n}} \mathsf{g}(\Gamma^2)'(\zeta)d\zeta\Big| \leq Ac(n)r_{k,n} \leq k\frac{A}{1-A}\,.$$

Therefore, setting

$$c_+(A) = (1+A)\exp\Big[\frac{2A}{1-A}\Big]\,,$$

one gets (see (4.22))

$$\begin{aligned} I^d_{k,n}(\Gamma^2) &\leq \frac{\sqrt{1+A}}{\sqrt{k}} c_+^k \, c(n)^k \, \frac{k!\, \mathrm{e}^k}{k^k} \, \phi^*_\Lambda(\Gamma^2)^n \\ &\simeq \sqrt{2\pi(1+A)} c_+^k \, c(n)^k \, \phi^*_\Lambda(\Gamma^2)^n \to \sqrt{2\pi} c(n)^k \, \phi^*_\Lambda(\Gamma^2)^n \quad \text{as } A \to 0. \end{aligned}$$

Recall that $\phi^*_\Lambda(\Gamma^2)^n$ verifies inequalities (4.21), that is, if β is large,

$$\phi^*_\Lambda(\Gamma^2)^n \approx \mathrm{e}^{-\beta n \|\Gamma^2\|}\,.$$

Similarly, if

$$c_-(A) = (1-A)\exp\Big[-\frac{2A}{1-A^2}\Big]\,,$$

then

$$I^d_{k,n}(\Gamma^2) \geq \frac{\sqrt{2\pi(1-A)}}{10} c_-^k \, c(n)^k \, \phi^*_\Lambda(\Gamma^2)^n \to \frac{\sqrt{2\pi}}{10} c(n)^k \, \phi^*_\Lambda(\Gamma^2)^n \quad \text{as } A \to 0.$$

4.4. **Estimate of the k$^{\text{th}}$-derivative of the pressure.** One estimates the derivative $[g^2_\Lambda]^{(k)}_{\mu^*}$ for large enough k and β. The result is formulated in proposition 4.2.

The parameters θ and A are not yet fixed. Two new parameters are introduced, ε' and η. It is important to see that one can choose these parameters in a consistent manner. Let $0 < \theta < 1$, $A \leq 1/25$, and set

$$\hat{\theta} := \theta(1 - 2\sqrt{A})\,.$$

Let $\varepsilon' > 0$ and χ_2' so that

$$(1+\varepsilon')\chi_2' > \chi_2(\infty)\,. \tag{4.25}$$

I fix the values of θ, and ε' by the following conditions. I choose $0 < A_0 < 1/25$; the parameters θ and ε' are chosen so that

$$\mathrm{e}^{\frac{1}{d}} \, \frac{1}{\theta(1-2\sqrt{A_0})} < \frac{d}{d-1} \, \frac{c_-(A_0)^{\frac{d-1}{d}}}{1+\varepsilon'} \quad \text{and} \quad \frac{1-2\sqrt{A_0}}{1+\varepsilon'} \, \frac{d}{d-1} > 1\,. \tag{4.26}$$

This is possible, because

$$\frac{d}{(d-1)} > \mathrm{e}^{\frac{1}{d}}\,.$$

Given θ, the value of $\theta^\star$ is fixed in proposition 4.1. From now on the values of θ, $\theta^\star$ and ε' are fixed once for all.

Notice that conditions (4.26) are still satisfied with the same values of θ and ε' if one replaces in (4.26) A_0 by $0 < A < A_0$. This means that one is still free to choose $A < A_0$. A is fixed in subsection 4.4.2.

Given k, there is a natural distinction between contours Γ^2, according to the fact that $k \geq k_+(\Gamma^q|\theta, A, \beta)$, or $k < k_+(\Gamma^q|\theta, A, \beta)$ (see (4.24)).

Definition 4.1. *A contour Γ^q is a*

(1) k-small contour, *if $\hat{\theta}\beta V(\Gamma^q) R_q(V(\Gamma^q)) \leq k$;*
(2) fat contour, *if for $\eta \geq 0$, $V(\Gamma^q)^{\frac{d-1}{d}} \leq \eta \, \|\Gamma^q\|$;*
(3) k-large and thin contour, *if $\hat{\theta}\beta V(\Gamma^q) R_q(V(\Gamma^q)) > k$, $V(\Gamma^q)^{\frac{d-1}{d}} > \eta \, \|\Gamma^q\|$.*

The parameter η is fixed in subsection 4.4.1.

I now make precise what I mean by k *large enough*. The answer depends on the parameters A, η and β.

By lemma 4.1 $V \mapsto V R_2(V)$ is increasing in V, and there exists $N(\chi_2')$ such that

$$R_2(V) \geq \frac{\chi_2'}{V^{\frac{1}{d}}} \quad \text{if} \quad V \geq N(\chi_2') \,. \tag{4.27}$$

The first condition is that there are k-small contours, which have a large volume. Precisely, one assumes that there is a k-small contour Γ^2 such that $V(\Gamma^2) \geq N(\chi_2')$. The second condition is that one can apply the stationary phase analysis for the large and thin contours. Therefore, one assumes (see lemma 4.2 and (4.20)) that $k > k_0(A)$, and that for a k-large and thin contour inequalities (4.28) are verified,

$$C_8 \frac{|\Gamma^2|}{V(\Gamma^2)} \leq \frac{C_8}{\rho\eta V(\Gamma^2)^{\frac{1}{d}}} \leq \frac{A}{2} \,. \tag{4.28}$$

Moreover, one assumes that β is large enough, so that (see (4.20) and (4.28))

$$\Big|\frac{d}{d\zeta}\mathsf{g}(\Gamma^2)(\zeta)\Big| \leq A \,.$$

The third condition is similar[44] to (4.28). It is assumed to be satisfied in order to control the large and thin contours. These conditions imply that there exists $K(A, \eta, \beta) < \infty$ such that if $k \geq K(A, \eta, \beta)$, then k is *large enough*. From now on $k \geq K(A, \eta, \beta)$.

4.4.1. *Contribution to $[g_\Lambda^q]_{\mu^*}^{(k)}$ from the k-small and fat contours.* Let Γ^2 be a k-small contour. Since $V \mapsto R_2(V)$ is decreasing in V, $u_\Lambda(\Gamma^2)$ is analytic in the region

$$\{z : \, \mathrm{Re} z \leq \mu^*(\mathrm{Im} z; \beta) + \theta \Delta^{-1} R_2(V^*)\} \cap \mathbb{U}_0 \,,$$

where V^* is the maximal volume of k-small contours. V^* satisfies

$$V^{*\frac{d-1}{d}} \leq \frac{k}{\hat{\theta}\beta\chi_2'} \,.$$

[44]Condition (2.39) in [FrPf1],

$$\frac{C_1}{\rho\Delta(1 - A_0)\eta V(\Gamma^2)^{\frac{1}{d}}} \leq \frac{1}{10} \,.$$

.

Hence (see (4.27))

$$\theta\Delta^{-1}R_2(V^*) \geq \hat\theta\Delta^{-1}\chi_2' V^{*-\frac{1}{d}} \geq \Delta^{-1}\big(\hat\theta\chi_2'\big)^{\frac{d}{d-1}}\beta^{\frac{1}{d-1}}k^{-\frac{1}{d-1}}\,.$$

One estimates the derivative of $u_\Lambda(\Gamma^2)$ by the Cauchy formula with a disc of center μ^* and radius $\Delta^{-1}\big(\hat\theta\chi_2'\big)^{\frac{d}{d-1}}\beta^{\frac{1}{d-1}}k^{-\frac{1}{d-1}}$. There exists a constant C_{10} such that

$$\Big|\sum_{\substack{\Gamma^2:\mathrm{Int}\,\Gamma^2\ni 0\\ \Gamma^2\ k\text{-small}}}[u_\Lambda(\Gamma^2)]^{(k)}_{\mu^*}\Big| \leq C_{10}\Big(\frac{\Delta}{\beta^{\frac{1}{d-1}}(\hat\theta\chi_2')^{\frac{d}{d-1}}}\Big)^k k!\,k^{\frac{k}{d-1}}\,.$$

If one chooses η small enough, then the contribution of fat contours (which are not k-small) is negligible compare to the contribution of the small contours. One fixes the value of the parameter η, so that this is the case.

4.4.2. *Contribution to $[g^q_\Lambda]^{(k)}_{\mu^*}$ from the k-large and thin contours.* The k-large and thin contours are the important contours. For them one has lower and upper bounds for $[\phi_\Lambda(\Gamma^2)^n]^{(k)}_{\mu^*}$. Using these bounds one gets upper and lower bounds on $-[u_\Lambda(\Gamma^2)]^{(k)}_{\mu^*}$. There are two cases.

I. Assume that $R_1(V(\Gamma^2)) \geq R_2(V(\Gamma^2))$, or that $V(\Gamma^2)$ is so large that

$$\hat\theta\beta V(\Gamma^2)R_1(V(\Gamma^2)) > k\,.$$

Under these conditions one can apply lemma 4.2 with a disc $D_{r_{k,n}}$ so that $\partial D_{r_{k,n}} = \partial D^d_{r_{k,n}}$. Indeed, either $R_1(V(\Gamma^2)) \geq R_2(V(\Gamma^2))$, and then one applies lemma 4.2 with $R = \theta\Delta^{-1}R_2(V(\Gamma^2))$, or this is not true, but the other condition is valid, so that one chooses $R = \theta\Delta^{-1}R_1(V(\Gamma^2))$. In both cases $r_{k,n} < R$, which implies $\partial D_{r_{k,n}} = \partial D^d_{r_{k,n}}$.
From the estimates for $[\phi_\Lambda(\Gamma^2)^n]^{(k)}_{\mu^*}$, one obtains estimates on $[u_\Lambda(\Gamma^2)]^{(k)}_{\mu^*}$:
there exists a function $D(k)$, $\lim_{k\to\infty}D(k) = 0$, such that for β sufficiently large and A sufficiently small,

$$(1-D(k))\,[\phi_\Lambda(\Gamma^2)]^{(k)}_{\mu^*} \leq -[u_\Lambda(\Gamma^2)]^{(k)}_{\mu^*} \leq (1+D(k))\,[\phi_\Lambda(\Gamma^2)]^{(k)}_{\mu^*}\,.$$

II. The second case is when

$$\hat\theta\beta V(\Gamma^2)R_1(V(\Gamma^2)) \leq k \leq \hat\theta\beta V(\Gamma^2)R_2(V(\Gamma^2))\,.$$

Since the contours are also thin,

$$\begin{aligned}\beta\|\Gamma^2\| &\leq \eta^{-1}\hat\theta^{-1}\chi_1(1)^{-1}\beta\hat\theta\chi_1(1)V(\Gamma^2)^{\frac{d-1}{d}}\\ &\leq \eta^{-1}\hat\theta^{-1}\chi_1(1)^{-1}\beta\hat\theta V(\Gamma^2)R_1(V(\Gamma^2))\\ &\leq \eta^{-1}\hat\theta^{-1}\chi_1(1)^{-1}k \equiv \lambda k\,.\end{aligned} \tag{4.29}$$

One chooses $R = \beta\Delta^{-1}R_2(V(\Gamma^2))$ in lemma 4.2. The integration in (4.15) is decomposed into two parts (see figure 2), and one shows that the contribution from the integration over $\partial D^g_{r_{k,n}}$ is negligible for large enough β. At that point one uses the fact that one can choose A small. This fixes the value of A. Inequality (4.29) is crucial, because it implies that the contribution from the integration over $\partial D^d_{r_{k,n}}$ is not too small, because the surface energy of a contour is not too large.

Lemma 4.3. *There exists $0 < A' \leq A_0$ such that for all β sufficiently large, the following holds. If $k \geq K(A', \eta, \beta)$ and Γ^2 is a k-large and thin contour, then*

$$-[u_\Lambda(\Gamma^2)]^{(k)}_{\mu^*} \geq \frac{1}{20}(1 - D(k))\big(\beta\Delta V(\Gamma^2)\big)^k c_-^k\, \phi^*_\Lambda(\Gamma^2)\,.$$

Proposition 4.2. *There exists β' such that for all $\beta > \beta'$, the following holds. There exists an increasing diverging sequence $\{k_n\}$ such that for each k_n there exists $\Lambda(L_n)$ such that for all $\Lambda \supset \Lambda(L_n)$*

$$[g^2_\Lambda]^{(k_n)}_{\mu^*} \geq C_{14}^{k_n}\, k_n!^{\frac{d}{d-1}}\, \Delta^{k_n} \beta^{-\frac{k_n}{d-1}}\, {\chi'_2}^{-\frac{dk_n}{d-1}}\,.$$

$C_{14} > 0$ is a constant independent of β, k_n and Λ.

Proof. One compares the contributions of the small and fat contours with that of the large and thin contours for $k \geq K(A', \eta, \beta)$. The contribution of the small and of the fat contours is at most

$$C_{10}\Big(\Delta\, \beta^{-\frac{1}{d-1}}\, {\chi'_2}^{-\frac{d}{d-1}}\Big)^k \Big(\frac{\mathrm{e}^{\frac{1}{d}}}{\theta(1-\sqrt{A'})}\Big)^{k\frac{d}{d-1}} k!^{\frac{d}{d-1}}\,.$$

The contribution to $[g^2_\Lambda]^{(k)}_{\mu^*}$ of each large and thin contour is nonnegative. By assumption (4.25) and the definition of the isoperimetric constant χ_2, there exists a sequence Γ^2_n, $n \geq 1$, such that

$$\lim_{n\to\infty} \|\Gamma^2_n\| \to \infty \quad \text{and} \quad V(\Gamma^2_n)^{\frac{d-1}{d}} \geq \frac{\|\Gamma^2_n\|}{(1+\varepsilon')\chi'_2}\,.$$

Since $x^{k\frac{d}{d-1}}\mathrm{e}^{-x}$ has its maximum at $x = k\frac{d}{d-1}$, let

$$k_n := \Big\lfloor \frac{d-1}{d}\beta\|\Gamma^2_n\| \Big\rfloor\,.$$

For any n, Γ^2_n is a thin and k_n-large volume contour, since by (4.26)

$$\begin{aligned}\beta\,(1-2\sqrt{A'})V(\Gamma^2)R_2(V(\Gamma^2)) &\geq \beta\,(1-2\sqrt{A'})V(\Gamma^2)^{\frac{d-1}{d}}\chi'_2 \\ &\geq \frac{(1-2\sqrt{A'})}{1+\varepsilon'}\beta\|\Gamma^2_n\| \geq k_n\,.\end{aligned}$$

Let $\Lambda \supset \Gamma^2_n$. Using lemma 4.3 one shows that $-[u_\Lambda(\Gamma^2_n)]^{(k_n)}_{\mu^*}$ is bounded below by

$$C_{13}\Big(\Delta\, \beta^{-\frac{1}{d-1}}\, {\chi'_2}^{-\frac{d}{d-1}}\Big)^{k_n} \Big(\frac{d}{d-1}\frac{c_-(A')^{\frac{d-1}{d}}}{1+\varepsilon'}\,\mathrm{e}^{-O(\delta)}\Big)^{k_n\frac{d}{d-1}} k_n!^{\frac{d}{d-1}}\,.$$

By the choice (4.26), if δ is small enough, i.e. β large enough, then

$$\frac{\mathrm{e}^{\frac{1}{d}}}{\theta(1-2\sqrt{A'})} < \frac{d}{d-1}\,\frac{c_-(A')^{\frac{d-1}{d}}}{1+\varepsilon'}\mathrm{e}^{-O(\delta)}\,.$$

Hence the contribution of the small and fat contours is negligible for large k_n. Let $\Lambda(L_n)$ be a box which contains at least $|\Lambda(L_n)|/4$ translates of Γ^2_n. For any $\Lambda \supset \Lambda(L_n)$, if k_n and β are large enough, then there exists a constant $C_{14} > 0$, independent of β, k_n and Λ, such that

$$[g^2_\Lambda]^{(k_n)}_{\mu^*} \geq C_{14}^{k_n}\, k_n!^{\frac{d}{d-1}}\, \Delta^{k_n} \beta^{-\frac{k_n}{d-1}}\, {\chi'_2}^{-\frac{dk_n}{d-1}}\,.$$

□

5. Non-analyticity and the van der Waals limit

This section is devoted to an exposition of the main results of Friedli's PhD thesis [Fr]. I do not give all details, since they can be found in [FrPf2] or [Fr], and because several arguments are similar to those of section 4.

It is important to understand *how the breakdown of analyticity at a first order phase transition point relates to the range of interaction, and how it is restored in the mean field limit*. This natural and pertinent question has been formulated and studied for the first time, as far as I know, by Sacha Friedli, who investigated the ferromagnetic Kac-Ising model in the van der Waals limit. This limit gives a way of interpolating finite range interaction systems and mean field models. The scaling parameter $0<\gamma<1$ which is used in the van der Waals limit is directly related to the inverse range of the interaction. For any γ the range of the interaction is finite, and the interaction of a given spin with all other spins remains uniformly bounded in γ. From theorem 2.1 one deduces the existence of $\beta_0(\gamma)$ so that for any $\beta>\beta_0(\gamma)$ there is no analytic continuation of the pressure at a first order phase transition. However, the validity of this result, with respect to the temperature, is not good enough if one wants to take the limit $\gamma\to 0$, because $\lim_\gamma \beta_0(\gamma)=\infty$. Sacha Friedli proved that *there exists a temperature $\beta_\star$*, independent *of γ, and γ_0, so that for any $\beta\geq\beta_\star$ and any $0<\gamma\leq\gamma_0$ the pressure p_γ has no analytic continuation at the first order phase transition point $h=0$. Furthermore, there exists also a constant $C=C(\beta)$*, independent of γ, *so that*

$$\left|p_\gamma^{(k)}(0^\pm)\right|\leq C^k k! \quad \text{for all } k\leq k_1(\gamma), \text{ with } k_1(\gamma)=\gamma^{-d}.$$

Thus, for the Kac-Ising ferromagnet on $\mathbb{Z}^d$ ($d\geq 2$) at low temperature, the pressure has no analytic continuation at the transition point *as long as the range of interaction is finite* ($\gamma>0$). Analytic continuation occurs only *after* the van der Waals limit ($\gamma\to 0$). One can prove similar results concerning the free energy f_γ for given magnetization m, which is related to the pressure p_γ by a Legendre transformation. In the lattice gas interpretation of the model, this thermodynamic potential f_γ is the (Helmholtz) free energy for given particle density. It is a convex function of m, and in the van der Waals limit $f_0(m):=\lim_{\gamma\to 0} f_\gamma(m)$ is the convex envelope of the mean field free energy $f_{\mathrm{mf}}(m)$ (see theorem 1.1),

$$f_{\mathrm{mf}}(m)=-\frac{1}{2}m^2-\frac{1}{\beta}I(m) \quad \text{with } m\in[-1,+1]\,. \tag{5.1}$$

In this formula $I(m)$ is the entropy term,

$$I(m):=-\frac{1-m}{2}\ln\frac{1-m}{2}-\frac{1+m}{2}\ln\frac{1+m}{2}\,.$$

When $\beta\leq 1$, f_{mf} is a strictly convex function of m, but when $\beta>1$, f_0 has a plateau for $m\in[-m^*(\beta),+m^*(\beta)]$, where $m^*(\beta)$ is the positive solution of the mean field equation $m=\tanh(\beta m)$. f_0 is analytic on $(-1,-m^*(\beta))$ and $(+m^*(\beta),+1)$, and the analytic continuation of $f_0(m)$ beyond $\pm m^*(\beta)$ is given by the mean field free energy (5.1). *After* the van der Waals limit, one has the same situation as in the van der Waals-Maxwell theory.

The main difficulty is to prove the existence of $\beta_\star$. It is necessary to study the model on a coarse-grained scale, related to the range γ^{-1} of the interaction. The coarse-grained formulation of the model is based on a recent paper of Bovier and

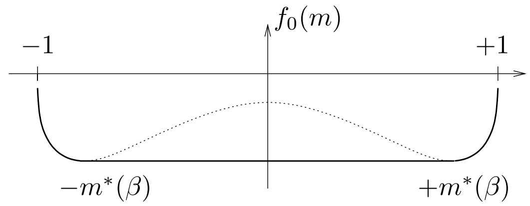

Figure 3: The free energy f_0 when $\beta > 1$. The dotted line is the analytic continuation provided by f_{mf}.

Zahradník [BoZ], in which this problem is solved. Once the coarse-grained description of the model is done, the rest of the analysis follows the same pattern as in section 4, and therefore I shall not expose it again. Due to the symmetry of the model, one knows that the phase transition point occurs at zero magnetic field. There is "no section 3" in this case. On the other hand, the contour models are more complicated because there are interactions between contours besides the basic hard-core condition. This also shows that theorem 2.1 can be proved in a more general context than that of section 2.

5.1. **Main results.** The model is a ferromagnetic Ising model with spin-variable $\sigma_i = \pm 1$ and interaction

$$J_\gamma(x) = c_\gamma \gamma^d \varsigma(\gamma x)\,,$$

with $0 < \gamma < 1$, and $\varphi : \mathbb{R}^d \to \mathbb{R}^+$ a function whose support is the cube $[-1,+1]^d$, and such that

$$\int \varsigma(x)\mathrm{d}x = 1\,.$$

The constant c_γ in the definition of the interaction is chosen so that

$$\sum_{x\in\mathbb{Z}^d:\, x\neq 0} J_\gamma(x) = 1\,.$$

The inverse of the scaling parameter γ is the range of the interaction. Spin configurations are denoted in this section by σ or η. Ω_Λ is the set of spin configurations in Λ and Ω the set of spin configurations on $\mathbb{Z}^d$. The restriction of a configuration σ to a subset $A \subset \mathbb{Z}^d$ is denoted by σ_A.

For a finite Λ and $\sigma \in \Omega_\Lambda$, the Kac-Ising hamiltonian is

$$H^h_\Lambda(\sigma) := -\sum_{\substack{\{i,j\}\subset\Lambda\\ i\neq j}} J_\gamma(i-j)\sigma_i\sigma_j - h\sum_{i\in\Lambda}\sigma_i\,,\ h\in\mathbb{R}\,.$$

The parameter h is the magnetic external field, and the magnetization in Λ is

$$m_\Lambda(\sigma) = \frac{1}{|\Lambda|}\sum_{i\in\Lambda}\sigma_i \in [-1,+1]\,.$$

The canonical partition function is

$$Z(\Lambda,m) := \sum_{\substack{\sigma_\Lambda\in\Omega_\Lambda:\\ m_\Lambda(\sigma)=m}} \exp\big(-\beta H^0_\Lambda(\sigma_\Lambda)\big)\,,$$

and the free energy for given magnetization m

$$f_\gamma(m) := -\lim_{\Lambda\uparrow\mathbb{Z}^d} \frac{1}{\beta|\Lambda|} \ln Z(\Lambda, m(\Lambda)) \,.$$

In the definition of f_γ the thermodynamic limit $\Lambda \uparrow \mathbb{Z}^d$ is taken along a sequence of cubes, and the sequence $m(\Lambda)$ is such that $m(\Lambda) \to m$.

There are few technical restrictions for the function ς, which can be found in [Fr]. In these lectures I always consider a specific and convenient choice,

$$\varsigma(x) := \begin{cases} 2^{-d} & \text{if } x \in [-1,1]^d; \\ 0 & \text{otherwise.} \end{cases}$$

In this setting, the main result for the free energy f_γ is the following theorem.

Theorem 5.1. *There exists $\beta_\star$ and $\gamma_0 > 0$ such that for all $\beta \geq \beta_\star$, $\gamma \in (0, \gamma_0)$, f_γ is analytic at any $m \in (-1, +1)$, except at $\pm m^*(\beta, \gamma)$, where*

$$m^*(\beta, \gamma) := p_\gamma^{(1)}(0^+) \,.$$

f_γ has no analytic continuation beyond $-m^*(\beta, \gamma)$ along the real path $m < -m^*(\beta, \gamma)$. f_γ has no analytic continuation beyond $m^*(\beta, \gamma)$ along the real path $m > m^*(\beta, \gamma)$.

This result is in favor of the idea that *finiteness of the range of interaction is responsible for absence of analytic continuation.*

The proof of theorem 5.1 is obtained by working in the more appropriate grand canonical ensemble (in the lattice gas terminology), in which the constraint on the magnetization is replaced by a magnetic field. Let

$$Z(\Lambda) := \sum_{\sigma \in \Omega_\Lambda} \exp\big(-\beta H_\Lambda^h(\sigma)\big) \,.$$

As before, the pressure is

$$p_\gamma(h) := \lim_{\Lambda\uparrow\mathbb{Z}^d} p_{\gamma,\Lambda}(h) \quad \text{with} \quad p_{\gamma,\Lambda}(h) := \frac{1}{\beta|\Lambda|} \ln Z(\Lambda) \,.$$

The free energy and pressure are related by a Legendre transform:

$$f_\gamma(m) = \sup_{h\in\mathbb{R}} (hm - p_\gamma(h)) \,.$$

The analytic properties of f_γ at $\pm m^*(\beta, \gamma)$ will be obtained from those of p_γ at $h = 0$. By the theorem of Lee and Yang [LeY], p_γ is analytic in the complex plane except on the imaginary axis.

Theorem 5.2. *There exists $\beta_\star$, $\gamma_0 > 0$ and a constant $C_r > 0$ such that for all $\beta \geq \beta_\star$, $\gamma \in (0, \gamma_0)$, the following holds:*
(1) The pressure p_γ is C^∞ at $0^\pm$. There exists a constant $C_+ > 0$ such that for all $k \in \mathbb{N}$,

$$|p_\gamma^{(k)}(0^\pm)| \leq \big(C_+ \gamma^{\frac{d}{d-1}} \beta^{-\frac{1}{d-1}}\big)^k k!^{\frac{d}{d-1}} + C_r^k k! \,.$$

(2) The pressure has no analytic continuation at $h = 0$. More precisely, there exists $C_- > 0$ and an unbounded increasing sequence of integers $k_1, k_2, \ldots$ such that for all $k \in \{k_1, k_2, \ldots\}$,

$$|p_\gamma^{(k)}(0^\pm)| \geq \big(C_- \gamma^{\frac{d}{d-1}} \beta^{-\frac{1}{d-1}}\big)^k k!^{\frac{d}{d-1}} - C_r^k k! \,.$$

The lower bound in theorem 5.2 becomes irrelevant when γ tends to 0. The integers k_i depend on γ and β, and

$$\lim_{\gamma\to 0} k_i = +\infty\,.$$

From the upper bound of theorem 5.2 it is easy to get

Corollary 5.1. *There exists $C = C(\beta)$ such that for small values of k, i.e. for $k \leq \gamma^{-d}$, the following upper bound is true,*

$$|p_\gamma^{(k)}(0^{\pm})| \leq C^k k!\,.$$

The crossover in the behaviour of the derivatives of the pressure is represented on figure 4.

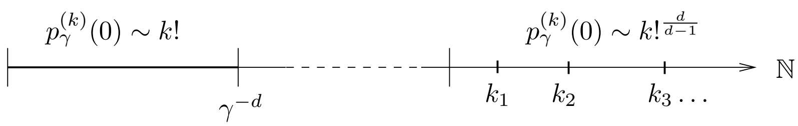

Figure 4: The derivatives of the pressure at $h = 0$, when $\gamma > 0$. The first ones ($k \leq \gamma^{-d}$) behave like those of an analytic function, but non-analyticity always dominates for large k.

Proof of theorem 5.1. Using the symmetry $p_\gamma(h) = p_\gamma(-h)$,

$$f_\gamma(m) = \sup_{h\geq 0}\big(hm - p_\gamma(h)\big)\,.$$

By the theorem of Lee and Yang, $h \mapsto p_\gamma(h)$ and $m \mapsto m_\gamma(h) := p_\gamma^{(1)}(h)$ are analytic in $\{\mathrm{Re} h > 0\}$. For all $m \in (m^*, 1)$,

$$f_\gamma(m) = h(m)m - p_\gamma(h(m))\,,$$

where $h_\gamma(m)$ is the unique solution of the equation $m = m_\gamma(h)$. If $h \geq 0$, GKS inequalities imply

$$p_\gamma^{(2)}(h) = \beta \sum_{j\in\mathbb{Z}^d} \langle\sigma_0\sigma_j\rangle_h - \langle\sigma_0\rangle_h \langle\sigma_j\rangle_h \geq \beta\big(\,\langle\sigma_0\sigma_0\rangle_h - \langle\sigma_0\rangle_h \langle\sigma_0\rangle_h\,\big) = \beta\big(1 - \langle\sigma_0\rangle_h^2\,\big)\,.$$

Since $p_\gamma^{(2)}(h) \neq 0$ for all $h > 0$, the biholomorphic mapping theorem[45] implies that $m \mapsto h_\gamma(m)$ is analytic in a complex neighbourhood of each $m \in (m^*, 1)$. So f_γ, which is a composition of analytic maps, is analytic on $(m^*, 1)$.

Proof that f_γ has no analytic continuation at m^*. Assume this is wrong.

$$h_\gamma^{(1)}(m^*) = \lim_{m\searrow m^*} h_\gamma^{(1)}(m) = \lim_{h\searrow 0} m_\gamma^{(1)}(h)^{-1} = \lim_{h\searrow 0} p_\gamma^{(2)}(h)^{-1} \neq 0\,, \tag{5.2}$$

since $p_\gamma^{(2)}(0^+)$ is bounded at $h = 0$. Again, (5.2) implies that the inverse of $h_\gamma = h_\gamma(m)$ can be inverted in a neighbourhood of m^* and that the inverse, $m_\gamma = m_\gamma(h)$, is analytic at $h = 0$. This is a contradiction with theorem 5.2. $\square$

[45]Let $g : D \to \mathbb{C}$ be an analytic function, $z_0 \in D$ be a point such that $g'(z_0) \neq 0$. Then there exists a domain $V \subset D$ containing z_0, such that the following holds: $V' = g(V)$ is a domain, and the map $g : V \to V'$ has an inverse $g^{-1} : V' \to V$ which is analytic, and which satisfies, for all $\omega \in V'$, ${g^{-1}}'(\omega) = \big(g'(g^{-1}(\omega))\big)^{-1}$. The proof of this result can be found in [Rem1], pp. 281-282.

5.2. **Coarse-grained description of the model.** In this subsection I write the model as a contour model with restricted phases. The contours are defined on a coarse-grained scale, which is of the order of the interaction range γ^{-1}. The Peierls condition is verified with a constant *independent of* γ (subsection 5.2). Then I consider the analysis of the restricted phases, which play the role of the ground-states in the standard the Pirogov-Sinai theory. All these three subsections are based on [FrPf2]. The proof of theorem 5.2 follows the pattern of the proof of theorem 2.1, with supplementary technical difficulties, since the fluctuations within the restricted phases induce many-body and long-range interactions among contours.

I first introduce some notations. Let $N \geq 1$;

$$B_N(x) := \{y \in \mathbb{Z}^d : |x-y| \leq N\} \quad \text{and} \quad B_N^\bullet(x) := B_N(x) \backslash \{x\}\,.$$

The N-neighbourhood of $\Lambda \subset \mathbb{Z}^d$ is

$$[\Lambda]_N := \bigcup_{x \in \Lambda} B_N(x)\,.$$

If $\sigma_\Lambda \in \Omega_\Lambda$, $\eta_{\Lambda^c} \in \Omega_{\Lambda^c}$, the concatenation $\sigma_\Lambda \eta_{\Lambda^c} \in \Omega$ is by definition:

$$(\sigma_\Lambda \eta_{\Lambda^c})_i = \begin{cases} (\sigma_\Lambda)_i & \text{if } i \in \Lambda\,, \\ (\eta_{\Lambda^c})_i & \text{if } i \in \Lambda^c\,. \end{cases}$$

The symbol # is used to denote either of the symbols + or −, or the constant configuration taking the value # at each site of $\mathbb{Z}^d$.

The interaction is rewritten as

$$J_\gamma(x) := \begin{cases} \dfrac{1}{|B^\bullet_{\gamma^{-1}}(0)|} & \text{if } 0 < |x| \leq \gamma^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

As in the Pirogov-Sinai theory, the first step is to define the notion of a correct point, respectively incorrect point of a spin configuration. A point j is $(\delta,+)$-correct for σ if in its γ^{-1}-neighbourhood there are not too many spins with value -1. The value of σ_j itself does not matter. Recall that γ^{-1} is the range of the interaction.

Definition 5.1. *Let $\delta \in (0,1)$, $\sigma \in \Omega$, $i \in \mathbb{Z}^d$.*

1. *i is* $(\delta,+)$-correct for σ *if* $|B^\bullet_{\gamma^{-1}}(i) \cap \{j : \sigma_j = -1\}| \leq \frac{\delta}{2}|B_{\gamma^{-1}}(i)|$.
2. *i is* $(\delta,-)$-correct for σ *if* $|B^\bullet_{\gamma^{-1}}(i) \cap \{j : \sigma_j = +1\}| \leq \frac{\delta}{2}|B_{\gamma^{-1}}(i)|$.
3. *i is* δ-correct for σ *if it is either $(\delta,+)$- or $(\delta,-)$-correct for σ.*
4. *i is* δ-incorrect for σ *if it is not δ-correct.*

If δ is sufficiently small, then in a γ^{-1}-neighbourhood of a $(\delta,+)$-correct point, all points are either $(\delta,+)$-correct or they are incorrect. So, $(\delta,+)$-correct points are distant from $(\delta,-)$-correct points.

Lemma 5.1. *Let $\delta \in (0, 2^{-d})$, $\sigma \in \Omega$. Then*
(1) If i is $(\delta,+)$-correct, the box $B_{\gamma^{-1}}(i)$ contains either $(\delta,+)$-correct or δ-incorrect points.
(2) If i is $(\delta,-)$-correct, the box $B_{\gamma^{-1}}(i)$ contains either $(\delta,-)$-correct or δ-incorrect points.

Proof. Let i be $(\delta,+)$-correct for σ, and $j \in B_{\gamma^{-1}}(i)$. Clearly

$$|B_{\gamma^{-1}}(i) \cap B_{\gamma^{-1}}(j)| \geq \frac{1}{2^d}|B_{\gamma^{-1}}(i)|\,.$$

Therefore, there are at least

$$\frac{1}{2^d}|B_{\gamma^{-1}}(i)| - \frac{\delta}{2}|B_{\gamma^{-1}}(i)| \geq \frac{1}{2}\frac{1}{2^d}|B_{\gamma^{-1}}(i)|$$

points, which are $(\delta,+)$-correct in $B_{\gamma^{-1}}(j)$, i.e. j cannot be $(\delta,-)$-correct. □

From now on δ is fixed in $(0, 2^{-d})$. The cleaned configuration $\overline{\sigma} \in \Omega$ is defined by

$$\overline{\sigma}_i := \begin{cases} +1 & \text{if } i \text{ is } (\delta,+) \text{ -correct for } \sigma, \\ -1 & \text{if } i \text{ is } (\delta,-) \text{ -correct for } \sigma, \\ \sigma_i & \text{if } i \text{ is } \delta\text{-incorrect for } \sigma. \end{cases}$$

For any set $M \subset \mathbb{Z}^d$, the partial cleaning $\sigma_M\overline{\sigma}_{M^c}$ coincides with σ on M and with $\overline{\sigma}$ on M^c. The cleaning and partial cleaning are always done according to the *original* configuration σ, with the fixed δ.

The set of δ-incorrect points of the configuration σ is denoted by $I_\delta(\sigma)$. The important property of the cleaning operation is that it can only change incorrect points for σ into correct points for the (partially) cleaned configuration.

Lemma 5.2. (I) *Let $M \subset \mathbb{Z}^d$. If i is $(\delta,+)$-correct for σ, then it remains $(\delta,+)$-correct for $\sigma_M\overline{\sigma}_{M^c}$. If i is $(\delta,-)$-correct for σ, then it remains $(\delta,-)$-correct for $\sigma_M\overline{\sigma}_{M^c}$.*
(II) *Let $M_1 \subset M_2$, $\delta' \in (0,\delta]$. Then $I_{\delta'}(\sigma_{M_1}\overline{\sigma}_{M_1^c}) \subset I_{\delta'}(\sigma_{M_2}\overline{\sigma}_{M_2^c})$.*

Proof. (I) If i is, say, $(\delta,+)$-correct for σ, then the cleaning of σ has the only effect, in the box $B_{\gamma^{-1}}(i)$, of changing some $-$ spins into $+$ spins (and never $+$ spins into $-$ spins). This is a consequence of lemma 5.1. Therefore the i remains $(\delta,+)$-correct for $\sigma_M\overline{\sigma}_{M^c}$.
(II) Let i be a $(\delta',+)$-correct point of $\sigma_{M_2}\overline{\sigma}_{M_2^c}$. One shows that it is also a $(\delta',+)$-correct point of $\sigma_{M_1}\overline{\sigma}_{M_1^c}$.

The two configurations $\sigma_{M_2}\overline{\sigma}_{M_2^c}$ and $\sigma_{M_1}\overline{\sigma}_{M_1^c}$ differ only on $M_2\backslash M_1$. Let $k \in M_2\backslash M_1$. There are three possibilities for the spin at k.
(1) If k is $(\delta,+)$-correct for σ then $\overline{\sigma}_k = +1$.
(2) If k is δ-incorrect for σ then $\overline{\sigma}_k = \sigma_k = (\sigma_{M_2}\overline{\sigma}_{M_2^c})_k$.
(3) The last possibility, a priori, is that k is $(\delta,-)$-correct for σ. By (I), if k is $(\delta,-)$-correct for σ, then k is also $(\delta,-)$-correct for $\sigma_{M_2}\overline{\sigma}_{M_2^c}$, and by lemma 5.1, i cannot be $(\delta,+)$-correct for $\sigma_{M_2}\overline{\sigma}_{M_2^c}$. This contradicts the fact that i is $(\delta',+)$-correct for $\sigma_{M_2}\overline{\sigma}_{M_2^c}$. Since only (1) and (2) occur, the lemma is proved. □

I now turn to the definition of the contours. Let $C^{(l)}$ be a partition of $\mathbb{Z}^d$ made of disjoint cubes of side length $l \in \mathbb{N}$, $l = \nu\gamma^{-1}$ with $\nu > 2$, and whose centers lie on the sites of a fixed sub-lattice of $\mathbb{Z}^d$. If $i \in \mathbb{Z}^d$, then $C_i^{(l)}$ is the unique element of the partition $C^{(l)}$, which contains the site i. The family of all subsets of $\mathbb{Z}^d$, which are unions of elements of $C^{(l)}$, is denoted by $\mathcal{L}^{(l)}$. For any set $A \subset \mathbb{Z}^d$, the thickening of A is

$$\{A\}_l := \bigcup_{i\in A} C_i^{(l)}\,.$$

As in the Pirogov-Sinai theory, contours are defined by δ-incorrect points. Since they are defined on the coarse-grained scale l, a possible definition of the boundary of a configuration would be

$$M' = \{[I_\delta(\sigma)]_{\gamma^{-1}}\}_l\,.$$

Notice that any $j \notin M'$ is either $(\delta,+)$-correct or $(\delta,-)$-correct. If the spin at $j \notin M'$ is $(\delta,+)$-correct ($(\delta,-)$-correct), then after cleaning it is a $+$-spin ($-$-spin). Moreover, by definition of correct/incorrect points, if $i \in M'$, with $|i-j| \leq \gamma^{-1}$, and if $j \notin M'$ is $(\delta,+)$-correct, then i is also $(\delta,+)$-correct. Of course, after cleaning outside M', i remains $(\delta,+)$-correct for the partially cleaned configuration. Unfortunately this is not strong enough, and one must require the stronger condition that i is $(\tilde\delta,+)$-correct, with $\tilde\delta < \delta$, for the partially cleaned configuration outside M'. Therefore, in order to define the notion of a boundary of a configuration, one introduces the family of subsets of $\mathcal{L}^{(l)}$ which have the desired properties, and then shows that this family of subsets is nonempty and stable for the intersection, so that one can define the boundary of a configuration as the smallest element of this family. The details are given in the next paragraph.

Let $\tilde\delta \in (0,\delta)$. For each $\sigma \in \Omega$ with $|I_{\tilde\delta}(\sigma)| < \infty$, let

$$\mathcal{E}(\sigma) := \left\{ M \in \mathcal{L}^{(l)} : M \supset [I_\delta(\sigma)]_{\gamma^{-1}},\, M \supset [I_{\tilde\delta}(\sigma_M \overline{\sigma}_{M^c})]_{\gamma^{-1}} \right\}.$$

(1) $\mathcal{E}(\sigma)$ is not empty. Indeed, let $M_0 := \{[I_{\tilde\delta}(\sigma)]_R\}_l$. If $M_0 = \emptyset$ then $I_{\tilde\delta}(\sigma) = I_\delta(\sigma) = \emptyset$ and any subset of $\mathbb{Z}^d$ is in $\mathcal{E}(\sigma)$. If $M_0 \neq \emptyset$, then $M_0 \in \mathcal{E}(\sigma)$, because $M_0 \in \mathcal{L}^{(l)}$, $M_0 \supset [I_{\tilde\delta}(\sigma)]_{\gamma^{-1}} \supset [I_\delta(\sigma)]_{\gamma^{-1}}$ and $M_0 \supset [I_{\tilde\delta}(\sigma)]_{\gamma^{-1}} \supset [I_{\tilde\delta}(\sigma_{M_0}\overline{\sigma}_{M_0^c})]_{\gamma^{-1}}$ by lemma 5.2.
(2) $\mathcal{E}(\sigma)$ is stable by intersection. Indeed, let $A, B \in \mathcal{E}(\sigma)$. Then clearly $A \cap B \supset [I_\delta(\sigma)]_R$; moreover, by lemma 5.2,

$$\begin{aligned} A &\supset [I_{\tilde\delta}(\sigma_A\overline{\sigma}_{A^c})]_R \supset [I_{\tilde\delta}(\sigma_{A\cap B}\overline{\sigma}_{(A\cap B)^c})]_R\,,\\ B &\supset [I_{\tilde\delta}(\sigma_B\overline{\sigma}_{B^c})]_R \supset [I_{\tilde\delta}(\sigma_{A\cap B}\overline{\sigma}_{(A\cap B)^c})]_R\,. \end{aligned}$$

Hence, one defines the boundary of the configuration σ as

$$I^*(\sigma) := \bigcap_{M\in\mathcal{E}(\sigma)} M\,.$$

The next property of $I^*(\sigma)$ is essential to prove the Peierls condition: there are sufficiently many $\tilde\delta$-incorrect points in $I^*(\sigma)$ for the partially cleaned configuration $\sigma_{I^*}\overline{\sigma}_{I^{*c}}$.

Lemma 5.3. *There exists, in the $2\gamma^{-1}$-neighbourhood of each box $C^{(l)} \subset I^*(\sigma)$, a point $j \in I^*(\sigma)$ which is $\tilde\delta$-incorrect for the configuration $\sigma_{I^*}\overline{\sigma}_{I^{*c}}$.*

Proof. Let $C^{(l)} \subset I^*(\sigma)$. First, suppose $I_\delta(\sigma) \cap [C^{(l)}]_{2\gamma^{-1}} \neq \emptyset$. Then each $j \in I_\delta(\sigma) \cap [C^{(l)}]_{2\gamma^{-1}}$ is δ-incorrect for σ, and hence $\tilde\delta$-incorrect for $\sigma_{I^*}\overline{\sigma}_{I^{*c}}$, since $\tilde\delta < \delta$ and σ and $\sigma_{I^*}\overline{\sigma}_{I^{*c}}$ coincide on $B_{\gamma^{-1}}(j)$.
Suppose[46] that $[I_\delta(\sigma)]_{\gamma^{-1}} \cap [C^{(l)}]_{\gamma^{-1}} = \emptyset$, and that the statement is wrong, i.e. $I_{\tilde\delta}(\sigma_{I^*}\overline{\sigma}_{I^{*c}}) \cap [C^{(l)}]_{2\gamma^{-1}} = \emptyset$. Then, set $I' := I^* \backslash C^{(l)}$ and show that $I' \in \mathcal{E}(\sigma)$, a contradiction with the definition of I^*. First, $I' \supset [I_\delta(\sigma)]_{\gamma^{-1}}$. Using Lemma 5.2, $I^* \supset [I_{\tilde\delta}(\sigma_{I^*}\overline{\sigma}_{I^{*c}})]_{\gamma^{-1}} \supset [I_{\tilde\delta}(\sigma_{I'}\overline{\sigma}_{I'^c})]_{\gamma^{-1}}$. Since $I_{\tilde\delta}(\sigma_{I^*}\overline{\sigma}_{I^{*c}}) \cap [C^{(l)}]_{2\gamma^{-1}} = \emptyset$ is

[46] Here I use the fact that $A \cap [B]_{2\gamma^{-1}} = \emptyset$ if and only if $[A]_{\gamma^{-1}} \cap [B]_{\gamma^{-1}} = \emptyset$.

equivalent to $[I_{\tilde{\delta}}(\sigma_{I^*}\overline{\sigma}_{I^{*c}})]_{\gamma^{-1}} \cap [C^{(l)}]_{\gamma^{-1}} = \emptyset$, this implies that $I' \supset [I_{\tilde{\delta}}(\sigma_{I'}\overline{\sigma}_{I'^c})]_{\gamma^{-1}}$, i.e. $I' \in \mathcal{E}(\sigma)$. □

Contrary to what happens in the standard theory of Pirogov-Sinai, it is less obvious to characterize the set of configurations which have the same boundary. Let

$$\mathcal{A}(\sigma) := \left\{\sigma' : \sigma'_{I^*(\sigma)} = \sigma_{I^*(\sigma)},\, I^*(\sigma') = I^*(\sigma)\right\}.$$

Let $\Lambda^\#(\sigma)$ be the set of points of $I^*(\sigma)^c$ that are $(\delta, \#)$-correct for σ. By lemma 5.1 $d(\Lambda^+(\sigma), \Lambda^-(\sigma)) > l$, and $\mathbb{Z}^d$ is partitioned into

$$\mathbb{Z}^d = I^*(\sigma) \cup \Lambda^+(\sigma) \cup \Lambda^-(\sigma).$$

Proposition 5.1. *$\mathcal{A}(\sigma) = \mathcal{D}(\sigma)$ if*

$$\mathcal{D}(\sigma) := \left\{\sigma' :\, \sigma'_{I^*(\sigma)} = \sigma_{I^*(\sigma)},\ \textit{each } i \in [\Lambda^\#(\sigma)]_{\gamma^{-1}} \textit{ is } (\delta,\#)\textit{-correct for } \sigma'\right\}.$$

Proof. Let $I^*(\sigma) \neq \emptyset$ (otherwise the statement is obvious).
(1) Assume $\sigma' \in \mathcal{A}(\sigma)$. Then $I^* \equiv I^*(\sigma) = I^*(\sigma') \supset [I_\delta(\sigma')]_{\gamma^{-1}}$, so that each $i \in [I^{*c}]_{\gamma^{-1}}$ is δ-correct for σ'. Let A be a maximal connected component of $[I^{*c}]_{\gamma^{-1}}$. There exists $i \in A$ such that $i \in I^*$, since by assumption $I^* \neq \emptyset$. By lemma 5.1, it suffices to show that i is $(\delta,+)$-correct for σ if and only if it is $(\delta,+)$-correct for σ'. Assume this is not the case, e.g. suppose i is $(\delta,+)$-correct for σ and $(\delta,-)$-correct for σ', i.e

$$|B^\bullet_{\gamma^{-1}}(i) \cap \{j :\, (\sigma_{I^*}\overline{\sigma}_{I^{*c}})_j = -1\}| \leq \frac{\tilde{\delta}}{2}|B^\bullet_{\gamma^{-1}}(i)|$$
$$|B^\bullet_{\gamma^{-1}}(i) \cap \{j :\, (\sigma'_{I^*}\overline{\sigma}'_{I^{*c}})_j = +1\}| \leq \frac{\tilde{\delta}}{2}|B^\bullet_{\gamma^{-1}}(i)|.$$

Since $i \in I^*$,

$$|B^\bullet_{\gamma^{-1}}(i) \cap I^{*c}| \leq (1 - 2^{-d})|B^\bullet_{\gamma^{-1}}(i)|.$$

Since $\sigma'_{I^*(\sigma)} = \sigma_{I^*(\sigma)}$, one gets a contradiction ($\tilde{\delta} < \delta < 2^{-d}$),

$$\begin{aligned}|B^\bullet_{\gamma^{-1}}(i)| &= |B^\bullet_{\gamma^{-1}}(i) \cap I^{*c}| + |B^\bullet_{\gamma^{-1}}(i) \cap \{j :\, (\sigma_{I^*}\overline{\sigma}_{I^{*c}})_j = -1\} \cap I^*| \\ &\quad + |B^\bullet_{\gamma^{-1}}(i) \cap \{j :\, (\sigma'_{I^*}\overline{\sigma}'_{I^{*c}})_j = +1\} \cap I^*| \\ &\leq (1 - 2^{-d} + \tilde{\delta})|B^\bullet_{\gamma^{-1}}(i)| < |B^\bullet_{\gamma^{-1}}(i)|.\end{aligned}$$

(2) Suppose $\sigma' \in \mathcal{D}(\sigma)$. Since σ' coincides with σ on $I^*(\sigma)$ and all points of $[I^*(\sigma)^c]_{\gamma^{-1}}$ are δ-correct for σ', $I_\delta(\sigma') = I_\delta(\sigma)$. Thus $I^*(\sigma) \supset [I_\delta(\sigma)]_{\gamma^{-1}} = [I_\delta(\sigma')]_{\gamma^{-1}}$. Then, since $\sigma_{I^*(\sigma)}\overline{\sigma}_{I^*(\sigma)^c} = {\sigma'}_{I^*(\sigma)}\overline{\sigma}'_{I^*(\sigma)^c}$, one has

$$I^*(\sigma) \supset [I_{\tilde{\delta}}(\sigma_{I^*(\sigma)}\overline{\sigma}_{I^*(\sigma)^c})]_{\gamma^{-1}} = [I_{\tilde{\delta}}({\sigma'}_{I^*(\sigma)}\overline{\sigma}'_{I^*(\sigma)^c})]_{\gamma^{-1}}.$$

Therefore $I^*(\sigma) \in \mathcal{E}(\sigma')$, i.e. $I^*(\sigma') \subset I^*(\sigma)$. Assume $I^*(\sigma)\backslash I^*(\sigma') \neq \emptyset$. Using the fact that σ and σ' coincide on $I^*(\sigma)\backslash I^*(\sigma')$, one has $\sigma_{I^*(\sigma')}\overline{\sigma}_{I^*(\sigma')^c} = \sigma'_{I^*(\sigma')}\overline{\sigma}'_{I^*(\sigma')^c}$. This gives, as above, $I^*(\sigma') \supset [I_{\tilde{\delta}}({\sigma'}_{I^*(\sigma')}\overline{\sigma}'_{I^*(\sigma')^c})]_{\gamma^{-1}} = [I_{\tilde{\delta}}(\sigma_{I^*(\sigma')}\overline{\sigma}_{I^*(\sigma')^c})]_{\gamma^{-1}}$. But $I^*(\sigma') \supset [I_\delta(\sigma')]_{\gamma^{-1}} = [I_\delta(\sigma)]_{\gamma^{-1}}$, so that $I^*(\sigma') \in \mathcal{E}(\sigma)$, i.e. $I^*(\sigma') \supset I^*(\sigma)$. Therefore $\sigma' \in \mathcal{A}(\sigma)$. □

Definition 5.2. *The connected components of the boundary $I^*(\sigma)$ are the* supports of the contours *of the configuration σ, and are written* $\operatorname{supp}\Gamma_1, \ldots, \operatorname{supp}\Gamma_n$. *A* contour *is a couple $\Gamma = (\operatorname{supp}\Gamma, \sigma_\Gamma)$, where σ_Γ is the restriction of σ to Γ.*

The notions of *label*, *external contour*, *interior of contour*, *compatibility of family of contours*, *boundary condition of a contour* are defined as in section 3. For each contour Γ with boundary condition $+$, there exists a unique configuration $\sigma[\Gamma]$, which coincides with σ_Γ on the $\operatorname{supp}\Gamma$, and which is equal to the labels of the components of $\mathbb{Z}^d \backslash \operatorname{supp}\Gamma$ otherwise. I also denote, as before, $\operatorname{supp}\Gamma$ by Γ when no confusion arise; in particular $|\Gamma| \equiv |\operatorname{supp}\Gamma|$. Notice also that the distance between the supports of two different contours of the same configuration is at least l.

5.3. **Proof of the Peierls condition.** Let $\Lambda \in \mathcal{L}^{(l)}$ be a finite set, $\sigma_\Lambda \in \Omega_\Lambda$ and set $\sigma := \sigma_\Lambda +_{\Lambda^c}$. Let

$$\phi_{ij}(\sigma_i,\sigma_j) := -\frac{1}{2} J_\gamma(i-j)(\sigma_i\sigma_j - 1) \quad \text{and} \quad \phi_{ij} := \phi_{ij}(+,-) \geq 0\,.$$

The hamiltonian with boundary condition $+_{\Lambda^c}$ is

$$H_\Lambda(\sigma) := H_\Lambda(\sigma_\Lambda +_{\Lambda^c}) = \sum_{\substack{\{i,j\}\cap\Lambda\neq\emptyset \\ i\neq j}} \phi_{ij}(\sigma_i,\sigma_j) + \sum_{i\in\Lambda} u(\sigma_i) \quad \text{with} \quad u(\sigma_i) = -h\sigma_i\,.$$

One identifies $I^*(\sigma)$ with $I^*(\sigma)\cap\Lambda$, and $\Lambda^\pm(\sigma)$ with $\Lambda^\pm(\sigma)\cap\Lambda$. The hamiltonian can be written in such a way that spins in regions $\Lambda^\#(\sigma)$ are subject to an effective external field $U^\#$.

The energy of the boundary of a configuration is by definition $H_{I^*}(\sigma_{I^*}\overline{\sigma}_{I^{*c}})$, and

$$H_{I^*}(\sigma_{I^*}\overline{\sigma}_{I^{*c}}) = \sum_\Gamma \Big(\|\Gamma\| + \sum_{i\in\Gamma} u(\sigma[\Gamma]_i) \Big)\,,$$

where the sum is over contours of the configuration σ. The surface energy of Γ is $\|\Gamma\|$; it is the same quantity as that of section 3.

Let $h = 0$ and $I^* = I^*(\sigma)$. $H_\Lambda(\sigma) - H_{I^*}(\sigma_{I^*}\overline{\sigma}_{I^{*c}})$ is equal to

$$\sum_{\#} \Big(\sum_{\{i,j\}\subset\Lambda^\#(\sigma)} \phi_{ij}(\sigma_i,\sigma_j) + \sum_{\substack{i\in\Lambda^\#(\sigma) \\ j\in I^*}} \phi_{ij}(\sigma_i,\sigma_j) + \sum_{\substack{i\in\Lambda^\#(\sigma) \\ j\notin\Lambda}} \phi_{ij}(\sigma_i,+) - \sum_{\substack{i\in\Lambda^\#(\sigma) \\ j\in I^*}} \phi_{ij}(\#,\sigma_j) \Big)\,.$$

Let $i \in \Lambda^+(\sigma)$. In the neighbourhood $B_{\gamma^{-1}}(i)$ of i, the majority of the spins are $+$-spins. There is an effective field acting on the spin at i, which is at first approximation equal to

$$\sum_{j\in B^\bullet_{\gamma^{-1}}(i)} \phi_{ij}(\sigma_i,+)\,.$$

Therefore, if i or $j \in \Lambda^+(\sigma)$, then it is natural to decompose $\phi_{ij}(\sigma_i,\sigma_j)$ as

$$\phi_{ij}(\sigma_i,\sigma_j) \equiv w^+_{ij}(\sigma_i,\sigma_j) + \phi_{ij}(\sigma_i,+) + \phi_{ij}(+,\sigma_j)\,.$$

Notice that

$$w^+_{ij}(\sigma_i,\sigma_j) := \phi_{ij}(\sigma_i,\sigma_j) - \phi_{ij}(\sigma_i,+) - \phi_{ij}(+,\sigma_j) = \begin{cases} -2\phi_{ij} & \text{if } \sigma_i = \sigma_j = -1 \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, if i or $j \in \Lambda^+(\sigma)$, then

$$w^-_{ij}(\sigma_i,\sigma_j) := \phi_{ij}(\sigma_i,\sigma_j) - \phi_{ij}(\sigma_i,-) - \phi_{ij}(-,\sigma_j) = \begin{cases} -2\phi_{ij} & \text{if } \sigma_i = \sigma_j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 5.4. *Define the potential*

$$U^{\#}(\sigma_i) := u(\sigma_i) + \sum_{j:j\neq i} \phi_{ij}(\sigma_i, \#) = -h\sigma_i + \sum_{j:j\neq i} \phi_{ij}(\sigma_i, \#)\,.$$

Then

$$H_\Lambda(\sigma) = H_{I^*}(\sigma_{I^*}\overline{\sigma}_{I^{*c}}) + \sum_{\#}\Big(\sum_{\substack{\{i,j\}\cap\Lambda^{\#}(\sigma)\neq\emptyset \\ i\neq j}} w^{\#}_{ij}(\sigma_i,\sigma_j) + \sum_{i\in\Lambda^{\#}(\sigma)} U^{\#}(\sigma_i)\Big)\,.$$

The central result of this section is proposition 5.2.

Proposition 5.2. *The surface energy satisfies the Peierls condition, i.e. there exists* $\rho = \rho(\tilde{\delta}, \nu) > 0$ *such that for all contours* Γ,

$$\|\Gamma\| \geq \rho|\Gamma|\,.$$

The constant ρ *is independent of* γ.

Remark. $|\Gamma|$ is the total number of lattice sites contained in the support of Γ. The support of a contour is a union of finitely many cubes of $\mathcal{L}^{(l)}$. So

$$|\Gamma| \geq (\nu\gamma^{-1})^d\,.$$

Another way of measuring the size of $\operatorname{supp}\Gamma$ would be to count the number of cubes $C^{(l)}$ contained in $\operatorname{supp}\Gamma$. In this case, the Peierls condition would become

$$\|\Gamma\| \geq \rho'\gamma^{-d}\#\{C^{(l)} \subset \operatorname{supp}\Gamma\}$$

(with a different constant ρ'), and $\beta\gamma^{-d}$ could be interpreted as an effective temperature for the system on the coarse-grained scale γ^{-1}.

Proof. One first shows that the effective field acting on a spin i is Lipschitz (with Lipschitz constant 2γ). Let $\sigma \in \Omega$, $i \in \mathbb{Z}^d$, $\# \in \{\pm\}$. Define

$$V_\sigma(i; \#) := \sum_{j:j\neq i} \phi_{ij}(\#, \sigma_j)\,.$$

Then, for $|x - y| \leq \gamma^{-1}$,

$$|V_\sigma(x;\#) - V_\sigma(y;\#)| \leq \gamma|x-y|\,. \tag{5.3}$$

Indeed, the difference $V_\sigma(x;\#) - V_\sigma(y;\#)$ is equal

$$\sum_{\substack{j\in B_{\gamma^{-1}}(x) \\ j\notin B_{\gamma^{-1}}(y)}} \phi_{xj}(\#,\sigma_j) + \sum_{j\in B_{\gamma^{-1}}(x)\cap B_{\gamma^{-1}}(y)} \big(\phi_{xj}(\#,\sigma_j) - \phi_{yj}(\#,\sigma_j)\big) - \sum_{\substack{j\in B_{\gamma^{-1}}(y) \\ j\notin B_{\gamma^{-1}}(x)}} \phi_{yj}(\#,\sigma_j)\,.$$

The middle sum vanishes; the first (last) sum can be estimated for $|x-y| \leq \gamma^{-1}$, by

$$\sum_{\substack{j\in B_{\gamma^{-1}}(x) \\ j\notin B_{\gamma^{-1}}(y)}} \phi_{xj}(\#,\sigma_j) \leq \frac{|B_{\gamma^{-1}}(x)| - |B_{\gamma^{-1}}(x)\cap B_{\gamma^{-1}}(y)|}{|B^{\bullet}_{\gamma^{-1}}(0)|} \leq \frac{|x-y|}{2\gamma^{-1}}\,.$$

By lemma 5.3 there exists in the $2\gamma^{-1}$-neighbourhood of each $C^{(l)} \subset \Gamma$ a point $j \in \Gamma$ which is $\tilde{\delta}$-incorrect for $\sigma[\Gamma]$. Let A be the set of all such points. One has $\Gamma \subset [A]_{l+2\gamma^{-1}}$. Let A_0 be any $4\gamma^{-1}$-approximant of A, that is $A_0 \subset A$, two points of

A_0 are at distance at least $4\gamma^{-1}$, and $A \subset [A_0]_{4\gamma^{-1}}$. Therefore $\Gamma \subset [A_0]_{l+6\gamma^{-1}}$. This implies that

$$|\Gamma| \leq |A_0|\,|B_{l+6\gamma^{-1}}(0)|\,. \tag{5.4}$$

Since each $i \in A_0$ is $\tilde{\delta}$-incorrect for $\sigma[\Gamma]$,

$$|B^{\bullet}_{\gamma^{-1}}(i) \cap \{k:\, \sigma[\Gamma]_k = +1\}| > \frac{\tilde{\delta}}{2}|B_{\gamma^{-1}}(i)| \quad (i \text{ is not } (\tilde{\delta},-)\text{-correct}),$$

and

$$|B^{\bullet}_{\gamma^{-1}}(j) \cap \{k:\, \sigma[\Gamma]_k = -1\}| > \frac{\tilde{\delta}}{2}|B_{\gamma^{-1}}(i)| \quad (i \text{ is not } (\tilde{\delta},+)\text{-correct}).$$

Hence, *independently of the value of* σ_i,

$$V_{\sigma[\Gamma]}(i;-) > \frac{\tilde{\delta}}{2} \quad \text{and} \quad V_{\sigma[\Gamma]}(i;+) > \frac{\tilde{\delta}}{2}\,. \tag{5.5}$$

One has

$$\begin{aligned}
\|\Gamma\| &\geq \frac{1}{2}\sum_{i\in A_0}\ \sum_{k\in B_{\gamma^{-1}}(i)\cap\Gamma}\ \sum_{l:l\neq k} \phi_{kl}(\sigma[\Gamma]_k, \sigma[\Gamma]_l)\\
&= \frac{1}{2}\sum_{i\in A_0}\ \sum_{k\in B_{\gamma^{-1}}(i)\cap\Gamma} V_{\sigma[\Gamma]}(k;\sigma[\Gamma]_k)\\
&\geq \frac{1}{2}\sum_{i\in A_0}\ \sum_{\substack{k\in B_{\gamma^{-1}}(i)\cap C_i^{(l)}\\ |k-i|\leq\frac{\tilde{\delta}}{4}\gamma^{-1}}} V_{\sigma[\Gamma]}(k;\sigma[\Gamma]_k)\,.
\end{aligned}$$

Using (5.3) and (5.5),

$$\begin{aligned}
V_{\sigma[\Gamma]}(k;\sigma[\Gamma]_k) &= V_{\sigma[\Gamma]}(i;\sigma[\Gamma]_k) + \big(V_{\sigma[\Gamma]}(k;\sigma[\Gamma]_k) - V_{\sigma[\Gamma]}(i;\sigma[\Gamma]_k)\big)\\
&\geq \frac{\tilde{\delta}}{2} - \gamma\,|k-i| \geq \frac{\tilde{\delta}}{4}\,.
\end{aligned}$$

From this one deduces the existence of $\rho > 0$, independent of $\gamma \in (0,\gamma_0)$, such that (see (5.4))

$$\|\Gamma\| \geq \frac{1}{2}|A_0|\,\frac{1}{2^d}\,|B_{\frac{\tilde{\delta}}{4}\gamma^{-1}}(0)|\,\frac{\tilde{\delta}}{4} \geq \frac{\tilde{\delta}}{2^{d+3}}\,|B_{l+6\gamma^{-1}}(0)|^{-1}|\Gamma|\,|B_{\frac{\tilde{\delta}}{4}\gamma^{-1}}(0)| \geq \rho|\Gamma|\,.$$

□

5.4. **Restricted phases.** The configurations of the restricted phases are those configurations such that either all points are $(\delta,+)$-correct or all points are $(\delta,-)$-correct. I consider the $+$ case, the other case is similar. I first define the kind of boundary conditions, which are admissible for the restricted partition functions associated with these phases. Let Λ be a finite subset in $\mathcal{L}^{(l)}$.

Definition 5.3. *A boundary condition $\eta_{\Lambda^c} \in \Omega_{\Lambda^c}$ is* $+$-admissible *if each $i \in [\Lambda]_{\gamma^{-1}}$ is $(\tilde{\delta},+)$-correct for the configuration $+_\Lambda\eta_{\Lambda^c}$.*

A $+$-admissible boundary condition means that, when looked from any point i inside of Λ, there is a majority of spins $+1$ on the boundary: for each $i \in [\Lambda]_{\gamma^{-1}}$,

$$|B^{\bullet}_{\gamma^{-1}}(i) \cap B(\eta_{\Lambda^c})| \leq \tfrac{\tilde{\delta}}{2}|B_{\gamma^{-1}}(i)|\,,$$

where

$$B(\eta_{\Lambda^c}) := \{i \in \Lambda^c : (\eta_{\Lambda^c})_i = -1\}\,.$$

Notice that the boundary condition specified by a contour on its interior is always admissible. Let $i \in [\Lambda]_{\gamma^{-1}}$, $\sigma_\Lambda \in \Omega_\Lambda$, and define

$$1_i(\sigma_\Lambda) := \begin{cases} 1 & \text{if } i \text{ is } (\delta,+)\text{-correct for } \sigma_\Lambda\eta_{\Lambda^c}\,,\\ 0 & \text{otherwise.}\end{cases}$$

($\eta_{\Lambda^c} \in \Omega_{\Lambda^c}$ is a $+$-admissible boundary condition.) The configuration which are allowed in a restricted phase are those verifying $1(\sigma_\Lambda) = 1$, with

$$1(\sigma_\Lambda) := \prod_{i\in[\Lambda]_{\gamma^{-1}}} 1_i(\sigma_\Lambda)\,.$$

Set $\sigma := \sigma_\Lambda\eta_{\Lambda^c}$. The hamiltonian for the restricted system is the one obtained in lemma 5.4 for a region of $+$-correct points. The restricted partition function with boundary condition η_{Λ^c} is

$$Z_r{}^{+}(\Lambda;\eta_{\Lambda^c}) := \sum_{\sigma_\Lambda\in\Omega_\Lambda} 1(\sigma_\Lambda)\exp\Big(-\beta\sum_{\substack{\{i,j\}\cap\Lambda\neq\emptyset\\ i\neq j}} w^{+}_{ij}(\sigma_i,\sigma_j) - \beta\sum_{i\in\Lambda}U^{+}(\sigma_i)\Big)\,.$$

One shows that $Z_r{}^{+}(\Lambda)$ can be put in the form

$$Z_r{}^{+}(\Lambda) = e^{\beta h|\Lambda|}\mathcal{Z}^{+}_r(\Lambda)\,,$$

where $\mathcal{Z}^{+}_r(\Lambda)$ is the partition function of a polymer model[47], having a normally convergent cluster expansion in the domain

$$H_+ = \left\{h \in \mathbb{C} : \mathrm{Re}h > -\tfrac{1}{8}\right\}.$$

The reason for $\log Z_r{}^{+}(\Lambda)$ to behave analytically at $h = 0$ is that the presence of contours is suppressed by $1(\sigma_\Lambda)$, and that on each spin $\sigma_i = -1$ acts an effective magnetic field

$$U^{+}(-1) = h + \sum_{j:j\neq i}\phi_{ij} = 1 + h\,,$$

which is close to 1 when h is in a neighbourhood of $h = 0$.

I now explain how one can express the restricted partition function $Z_r{}^{+}(\Lambda) \equiv Z_r{}^{+}(\Lambda;\eta_{\Lambda^c})$ as the partition function of a polymer model. Complete details are given in section 3 of [FrPf2].

The influence of a boundary condition can always be interpreted as a magnetic field acting on sites near the boundary. One rearranges the terms of the hamiltonian as follows:

$$\sum_{\substack{\{i,j\}\subset\Lambda\\ i\neq j}} w^{+}_{ij}(\sigma_i,\sigma_j) + \sum_{i\in\Lambda}\Big(U^{+}(\sigma_i) + \sum_{j\in\Lambda^c} w^{+}_{ij}(\sigma_i,(\eta_{\Lambda^c})_j)\Big)\,.$$

By defining a new effective non-homogeneous magnetic field

$$\mu^{+}_i(\sigma_i) := U^{+}(\sigma_i) + h + \sum_{j\in\Lambda^c} w^{+}_{ij}(\sigma_i,(\eta_{\Lambda^c})_j)\,,$$

[47] Representation of partition functions by polymer models was introduced by Kunz, [Ku1] and [GrKu], and [Ku2]. This is now a standard powerful method in statistical mechanics, which has been also used in constructive field theory and especially in lattice gauge theories with great success.

one can extract a volume term from ${Z_r}^+(\Lambda)$ and get ${Z_r}^+(\Lambda) = e^{\beta h|\Lambda|}\mathcal{Z}_r^+(\Lambda)$, where

$$\mathcal{Z}_r^+(\Lambda) := \sum_{\sigma_\Lambda \in \Omega_\Lambda} 1(\sigma_\Lambda) \exp\Big(-\beta \sum_{\substack{\{i,j\}\subset\Lambda \\ i\neq j}} w_{ij}^+(\sigma_i,\sigma_j) - \beta \sum_{i\in\Lambda} \mu_i^+(\sigma_i)\Big) .$$

Notice that the field $\mu_i^+(\sigma_i)$ becomes independent of η_{Λ^c} when $d(i,\Lambda^c) > \gamma^{-1}$. Since $w_{ij}^+(\sigma_i,\sigma_j) = 0$, if $\sigma_i = +1$ or $\sigma_j = +1$, and $\mu_i^+(+1) = 0$, only the spins σ_i with $\sigma_i = -1$ interact. The location of these spins are identified with the vertices of a graph. For each vertex of this graph one has a factor $e^{-\beta\mu_i^+(-1)}$. When $h \in H_+$,

$$\mathrm{Re}\mu_i^+(-1) = 1 + 2\mathrm{Re}h + \sum_{j\in\Lambda^c} w_{ij}^+(-,(\eta_{\Lambda^c})_j) \geq 1 - 2\tfrac{1}{8} - \tilde{\delta} > \tfrac{1}{2} ,$$

since $\tilde{\delta} < 2^{-d}$. The formulation of $\mathcal{Z}_r^+(\Lambda)$ in terms of polymers is a three step procedure. One first expresses $\mathcal{Z}_r^+(\Lambda)$ as a sum over graphs, satisfying a certain constraint inherited from $1(\sigma_\Lambda)$. Then, one associates to each graph a spanning tree and re-sum over all graphs having the same spanning tree. The weights of the trees have good decreasing properties. Finally, the constraint is expanded, yielding sets on which the constraint is *violated*. These sets are linked with trees. After a second partial re-summation, this yields a sum over polymers, which are nothing but particular graphs with vertices living on $\mathbb{Z}^d$ and whose edges are of length at most γ^{-1}.

I. Let $\mathcal{G}_\Lambda$ be the family of simple non-oriented graphs $G = (V,E)$ where $V \subset \Lambda$, each edge $e = \{i,j\} \in E$ has $d(i,j) \leq \gamma^{-1}$. For $e = \{i,j\}$, set $w_e^+ := w_{ij}^+(-,-)$. Notice that $w_e^+ = -2\phi_{ij} \leq 0$. Define also $\mu_i^+ := \mu_i^+(-1)$. Expanding the product over edges leads to the following expression

$$\mathcal{Z}_r^+(\Lambda) = \sum_{G\in\mathcal{G}_\Lambda} 1(V(G)) \prod_{e\in E(G)} (e^{-\beta w_e^+} - 1) \prod_{i\in V(G)} e^{-\beta\mu_i^+} ,$$

where $1(V) := 1(\sigma_\Lambda(V))$, and $\sigma_\Lambda(V) \in \Omega_\Lambda$ is defined by $\sigma_\Lambda(V)_i = -1$ if $i \in V$, $+1$ otherwise. In terms of graphs, the constraint $1(V(G)) = 1$ is satisfied if and only if

$$\sum_{\substack{e=\{i,j\} \\ j\in V(G)\cup B}} |w_e^+| \leq \delta , \quad \forall\, i \in [\Lambda]_{\gamma^{-1}} , \quad \text{where } B := B(\eta_{\Lambda^c}).$$

Moreover, the fact that the boundary condition η_{Λ^c} is $+$-admissible means that

$$\sum_{\substack{e=\{i,j\} \\ j\in B}} |w_e^+| \leq \tilde{\delta} , \quad \forall\, i \in [\Lambda]_{\gamma^{-1}} .$$

II. One chooses a deterministic algorithm[48] that assigns to each connected graph G_0 a spanning tree $T(G_0)$, in a translation invariant way (that is if G_0' is obtained from G_0 by translation then $T(G_0')$ is obtained from $T(G_0)$ by the same translation). The algorithm is applied to each component of each graph G appearing in the partition function. Let $\mathcal{T}_\Lambda \subset \mathcal{G}_\Lambda$ denote the set of all forests. Then

$$\mathcal{Z}_r^+(\Lambda) = \sum_{T\in\mathcal{T}_\Lambda} 1(V(T)) \prod_{\mathtt{t}\in T} \omega^+(\mathtt{t}) ,$$

[48]To be precise, one chooses the algorithm of chapter 3 of [Pf].

where the product is over trees of T, and the weight of each tree is defined by

$$\omega^+(\mathtt{t}) := \sum_{\substack{G\in\mathcal{G}_\Lambda:\\ T(G)=\mathtt{t}}} \prod_{e\in E(G)} (e^{-\beta w_e^+} - 1) \prod_{i\in V(G)} e^{-\beta\mu_i^+}\,.$$

Isolated sites $\{i\} \subset \Lambda$ are also considered as trees. In this case, $\omega^+(\{i\}) = e^{-\beta\mu_i^+}$.

Lemma 5.5. *Let $T \in \mathcal{T}_\Lambda$ be a forest such that $1(V(T)) = 1$. Then, uniformly in $h \in H_+$, for each tree $\mathtt{t} \in T$,*

$$|\omega^+(\mathtt{t})| \le \prod_{e\in E(\mathtt{t})} (e^{-\beta w_e^+} - 1) \prod_{i\in V(\mathtt{t})} e^{-\frac{1}{4}\beta}\,.$$

III. The constraint $1(V(T))$ depends on the relative positions of the trees. This "multi-body interaction" is treated by expanding

$$1(V(T)) = \prod_{i\in[\Lambda]_{\gamma-1}} 1_i(V(T)) = \prod_{i\in[\Lambda]_{\gamma-1}} (1 + 1_i^c(V(T))) = \sum_{M\subset[\Lambda]_{\gamma-1}} \prod_{i\in M} 1_i^c(V(T))\,,$$

where $1_i^c(V(T)) := 1_i(V(T)) - 1$. This yields

$$\mathcal{Z}_r^+(\Lambda) = \sum_{T\in\mathcal{T}_\Lambda} \sum_{M\subset[\Lambda]_{\gamma-1}} \Big(\prod_{i\in M} 1_i^c(V(T))\Big)\Big(\prod_{\mathtt{t}\in T} \omega^+(\mathtt{t})\Big)\,.$$

Consider a pair (T, M). Let $i \in M$. The function $1_i^c(V(T))$ is non-zero only when i is not $(\delta, +)$-correct; it depends on the presence of trees of T in the γ^{-1}-neighbourhood of i and possibly on the points of $B(\eta_{\Lambda^c})$ if $B_{\gamma^{-1}}(i)\cap\Lambda^c \neq \emptyset$. To make this dependence only local, one links the γ^{-1}-neighbourhoods of the points of M with the trees of T as follows.

(1) Let $N = N(M)$ be the graph whose vertices are given by

$$V(N) := \bigcup_{i\in M} B_{\gamma^{-1}}(i)\,.$$

There is an edge between two vertices of N, x and y, if and only if $\langle x, y\rangle$ is a pair of nearest neighbours of the same box $B_{\gamma^{-1}}(i)$ for some $i \in M$. The graph N decomposes naturally into connected components (in the sense of graph theory) $N_1, N_2, \dots, N_K$. Some of these components can intersect Λ^c.

(2) One links trees $\mathtt{t}_i \in T$ with components $N_j \in N$. To this end, one defines an abstract graph $\hat{G}$: to each tree $\mathtt{t}_i \in T$, one associates a vertex w_i, and to each component N_j one associates a vertex z_j. The edges of $\hat{G}$ are defined by the condition: $\hat{G}$ has only edges between vertices w_i and z_j, and this occurs if and only if $V(\mathtt{t}_i) \cap V(N_j) \neq \emptyset$. Consider a connected component of $\hat{G}$, whose vertices $\{w_{i_1}, \dots, w_{i_l}, z_{j_1}, \dots, z_{j_l}\}$ correspond to a set $P_l' = \{\mathtt{t}_{i_1}, \dots, \mathtt{t}_{i_l}, N_{j_1}, \dots, N_{j_l}\}$. One changes P_l' into a set P_l, using the following decimation procedure:

(a) if $P_l' = \{\mathtt{t}_{i_1}\}$ is a single tree then $P_l := P_l'$.

(b) if P_l' is not a single tree, then

(b_1) delete from P_l' all trees $\mathtt{t}_{i_k}$ that have no edges,

(b_2) for all trees $\mathtt{t}_{i_k}$ containing at least one edge, delete all edges $e \in E(\mathtt{t}_{i_k})$ whose both end-points lie in the same component N_{j_m}.

The resulting set is of the form $P_l = \{\mathtt{t}_{s_1}, \dots, \mathtt{t}_{s_l}, N_{j_1}, \dots, N_{j_l}\}$, where each tree $\mathtt{t}_{s_i}$ is a sub-tree of one of the trees $\{\mathtt{t}_{i_1}, \dots, \mathtt{t}_{i_l}\}$. P_l is called a polymer. The decimation procedure $P'_l \Rightarrow P_l$ is depicted on figure 5.

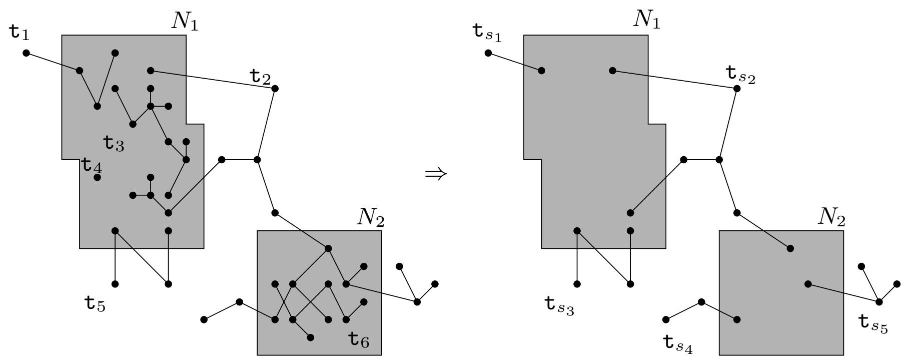

Figure 5: The decimation procedure $P'_l \Rightarrow P_l$. The hatched polygons represent the body $\mathcal{B}(P_l)$ and the legs are the trees $\{\mathtt{t}_{s_1}, \mathtt{t}_{s_2}, \mathtt{t}_{s_3}, \mathtt{t}_{s_4}, \mathtt{t}_{s_5}\}$. Each $\mathtt{t}_{s_j}$ is a sub-tree of some $\mathtt{t}_i$.

The body of P_l is $V(N_{j_1}) \cup \dots \cup V(N_{j_l})$; the legs of P_l are the trees $\{\mathtt{t}_{s_1}, \dots, \mathtt{t}_{s_l}\}$. A polymer can have no body (in which case it is a tree of $\mathcal{T}_\Lambda$), or no legs (in which case it is a single component N_{j_1}). The support $V(P)$ is the total set of sites:

$$V(P) := \bigcup_{\mathtt{t} \in \mathcal{L}(P)} V(\mathtt{t}) \cup \bigcup_i V(N_i)\,.$$

Often P also denotes $V(P)$. Two polymers are compatible if and only if $V(P_1) \cap V(P_2) = \emptyset$, denoted $P_1 \sim P_2$. Therefore, to each pair (T, M) there corresponds a family of pairwise compatible polymers $\{P\} := \varphi(T, M)$. The set of all possible polymers constructed in this way is denoted by $\mathcal{P}^+_\Lambda(\eta_{\Lambda^c})$. The representation of $\mathcal{Z}^+_r(\Lambda)$ in terms of polymers is then

$$\mathcal{Z}^+_r(\Lambda) \equiv \mathcal{Z}_r(\mathcal{P}^+_\Lambda(\eta_{\Lambda^c})) = \sum_{\substack{\{P\} \subset \mathcal{P}^+_\Lambda(\eta_{\Lambda^c}) \\ \text{compat.}}} \prod_{P \in \{P\}} \omega^+(P)\,,$$

where the weight is defined by

$$\omega^+(P) := \sum_{\substack{(T,M): \\ \varphi(T,M) = P}} \Big(\prod_{i \in M} 1^c_i(V(T)) \Big) \Big(\prod_{\mathtt{t} \in T} \omega^+(\mathtt{t}) \Big)\,.$$

The weight $\omega^+(P)$ depends on the position of P inside the volume Λ, via the boundary condition η_{Λ^c}.

Definition 5.4. *The* restricted pressures *are defined by*

$$p^\pm_{r,\gamma} := \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{\beta |\Lambda|} \log Z_r{}^\pm(\Lambda; \pm_{\Lambda^c})\;,$$

where the thermodynamic limit is taken along a sequence of cubes.

The next step is to prove that one can apply the cluster expansion to the polymer representation of the restricted partition function. I do not explain this technical part. One proves the next lemma[49].

Lemma 5.6. *There exists $c > 0$, so that given $\varepsilon > 0$, there exists $\beta(\varepsilon)$ with the property that*

$$\sum_{P:V(P)\ni 0} \sup_{\mathrm{Re}h>-1/8} |\omega^+(P)|\mathrm{e}^{c|V(P)|} \leq \varepsilon \quad \forall\, \gamma \in (0,\gamma_0)\,,$$

provided that $\beta \geq \beta(\varepsilon)$.

I state the main result concerning the restricted phases and their analyticity properties, for the case $\# = +$.

Theorem 5.3. *Let β be large enough, $\gamma \in (0,\gamma_0)$, $\Lambda \in \mathcal{L}^{(l)}$ and η_{Λ^c} be a $+$-admissible boundary condition. Then*

(1) $\mathcal{Z}_r(\mathcal{P}_\Lambda^+(\eta_{\Lambda^c}))$ *has a cluster expansion that converges normally in* $\{\mathrm{Re}h > -\frac{1}{8}\}$. *The maps $h \mapsto \log \mathcal{Z}_r(\mathcal{P}_\Lambda^+(\eta_{\Lambda^c}))$ and $h \mapsto p_{r,\gamma}^+(h)$ are analytic in* $\{\mathrm{Re}h > -\frac{1}{8}\}$.

(2) *There exists a function $\epsilon_r(\beta)$, verifying $\lim_{\beta\to\infty} \epsilon_r(\beta) = 0$, such that, uniformly in h for* $\mathrm{Re}h > -\frac{1}{8}$,

$$\big|\log \mathcal{Z}_r(\mathcal{P}_\Lambda^+(\eta_{\Lambda^c}))\big| \leq \epsilon_r(\beta)|\Lambda|\,.$$

(3) *Uniformly in h for* $\mathrm{Re}h > -\frac{1}{16}$,

$$\big|\frac{d}{dh}\log \mathcal{Z}_r(\mathcal{P}_\Lambda^+(\eta_{\Lambda^c}))\big| \leq \epsilon_r(\beta)|\Lambda|\,.$$

(4) *There exists a constant $C_r > 0$ such that for all integers $k \geq 2$,*

$$\frac{1}{|\Lambda|}\left|\frac{d^k}{dh^k}\ln {Z_r}^+(\Lambda;\eta_{\Lambda^c})\right|_{h=0} \leq C_r^k k!\,,\qquad |p_{r,\gamma}^{+(k)}(0)| \leq C_r^k k!\,.$$

The main part of the analysis of the model is done in a finite volume, say a (large) cubic box Λ in $\mathcal{L}^{(l)}$. Let $\gamma \in (0,\gamma_0)$. Let[50]

$$\Omega_\Lambda^+ := \{\sigma_\Lambda \in \Omega_\Lambda : d(I^*(\sigma_\Lambda +_{\Lambda^c}),\Lambda^c) > l\}\,.$$

One defines

$$Z^+(\Lambda) := \sum_{\sigma_\Lambda\in\Omega_\Lambda^+} e^{-\beta H_\Lambda(\sigma_\Lambda +_{\Lambda^c})}\,. \tag{5.6}$$

For each $\sigma_\Lambda \in \Omega_\Lambda^+$, the decomposition of $I^*(\sigma_\Lambda +_{\Lambda^c})$ into connected components yields an admissible family $\{\Gamma\}$, such that $\Gamma \subset \Lambda$ and $d(\Gamma,\Lambda^c) > l$ for each $\Gamma \in \{\Gamma\}$. Then, Λ is decomposed into $\Lambda = \{\Gamma\} \cup \Lambda^+ \cup \Lambda^-$, where $\Lambda^\#$ are the points of $\Lambda\backslash\{\Gamma\}$ that are $(\delta,\#)$-correct for the configuration $\sigma_\Lambda +_{\Lambda^c}$. If in (5.6) one sums over all configurations, which yield the same set of contours $\{\Gamma\}$, then one can write the partition function $Z^+(\Lambda)$ as

$${Z_r}^+(\Lambda;+_{\Lambda^c})\frac{\sum_{\{\Gamma\}\subset\Lambda}\Big(\prod_{\Gamma\in\{\Gamma\}}\rho(\Gamma)\Big){Z_r}^+(\Lambda^+;+_{\Lambda^c}\sigma_{\{\Gamma\}}){Z_r}^-(\Lambda^-;\sigma_{\{\Gamma\}})}{{Z_r}^+(\Lambda;+_{\Lambda^c})}\,, \tag{5.7}$$

[49]This is essentially lemma 3.5 in [Pf]. See also theorem 3.1 in [Pf].

[50]The condition $d(I^*(\sigma_\Lambda +_{\Lambda^c}),\Lambda^c) > l$ is convenient, but not really important. It is used in [FrPf2].

where the sum is over admissible families of contours, and

$$\rho(\Gamma) := e^{-\beta H_\Gamma(\sigma[\Gamma])}\,.$$

The restricted phases induce an interaction among the contours, via the polymers. One can write the partition function $Z^+(\Lambda)$ as

$$\mathrm{e}^{\beta h|\Lambda|}\,\mathcal{Z}_r^+(\Lambda)\,\Xi^+(\Lambda)\,, \tag{5.8}$$

where $\Xi^+(\Lambda)$ is the partition function of another polymer model, whose polymers are connected objects, which are made of contours and polymers describing the restricted phases. This is not immediate to obtain formula (5.8), but there is a well-known procedure for doing this, starting with formula (5.7). Once this is done, the analysis is similar to the analysis presented in section 4. Full details are given in section 4 of [FrPf2].

I end this section by some final remarks about the restricted pressures. From the full analysis one obtains the following expression for the pressure (see (5.8))

$$p_\gamma(h) = p^+_{r,\gamma}(h) + \mathrm{sing}^+_\gamma(h) \quad \text{if } h \geq 0. \tag{5.9}$$

The term $\mathrm{sing}^+_\gamma(h)$ is the contribution to the pressure, which is due to the presence of contours, and is defined using $\Xi^+(\Lambda)$ in (5.8). It is this term which is responsible for the absence of an analytic continuation of p_γ at $h=0$, because at $h=0$, the phase transition point, droplets of the $-$-phase, of arbitrary size, are stable. On the other hand, when γ is small, $\mathrm{sing}^+_\gamma(h)$ is small: if $h^+>0$, then there exist constants a and b, such that

$$|\mathrm{sing}^+_\gamma(h)| \leq a\mathrm{e}^{b\beta\gamma^{-d}} \quad \text{for all } h,\ 0\leq h\leq h^+.$$

This follows from proposition 5.2 (read also the remark following this proposition). In this sense the main contribution to the pressure is $p^+_{r,\gamma}$. $p^+_{r,\gamma}$ is the pressure of an *homogeneous* state characterized by the fact that all spins are $(\delta,+)$-correct.

By Hölder's inequality, the restricted pressure $p^+_{r,\gamma}$ is a convex function of h, $h>-1/8$. On $\mathrm{Re}h>-1/8$, as a consequence of Vitali theorem, the family of holomorphic functions $\{p^+_{r,\gamma}\}_\gamma$ is a normal family, which converges as $\gamma\to 0$. Furthermore,

$$\lim_{\gamma\to 0} p^+_{r,\gamma}(h) = p_{\mathrm{mf}}(h) \quad \text{if } h\geq 0.$$

Thus, the limiting function $\lim_{\gamma\to 0} p^+_{r,\gamma}(h)$ gives the analytic continuation of the mean-field pressure from $h>0$ to $-1/8<h<0$. Let

$$m^*_{r,\gamma} \equiv m^*_{r,\gamma}(\beta) := \frac{d}{dh}p^+_{r,\gamma}(h)\big|_{h=0}\,.$$

One has

$$\lim_{\gamma\to 0} m^*_{r,\gamma}(\beta,\gamma) = m^*_{\mathrm{mf}}\,,$$

where m^*_{mf} is the mean-field spontaneous magnetization. Using a Legendre transform one defines

$$f^+_{r,\gamma}(m) := \sup_{h\geq -1/8}\left(hm - p^+_{r,\gamma}(h)\right).$$

This defines a *convex* function on some interval $(m'_\gamma,1)$. Notice that, by definition of $m^*_{r,\gamma}$, $m\mapsto f^+_{r,\gamma}(m)$ has a minimum at $m^*_{r,\gamma}$. Moreover, by the fundamental theorem on Legendre transform of convex functions,

$$p^+_{r,\gamma}(h) := \sup_{m \in (m'_\gamma, 1)} \left(hm - f^+_{r,\gamma}(m) \right).$$

The part of the free energy $f^+_{r,\gamma}$ on $(m'_\gamma, m^*_{r,\gamma})$ can be interpreted as a metastable free energy, since it differs from the convex envelope of the mean-field free energy, which is the equilibrium free energy. For γ sufficiently small the functions $f^+_{r,\gamma}$ are defined on a common interval, say $(m', 1]$, such that $m' < m^*_{\mathrm{mf}}$. On that interval

$$\lim_{\gamma \to 0} f^+_{r,\gamma}(m) = f_{\mathrm{mf}}(m).$$

There are of course similar results for $p^-_{r,\gamma}$.

6. Conclusions

The main results, theorems 2.1, 5.1 and 5.2, are proven only at low temperatures and the proofs of these theorems used this fact heavily. However, the proofs of theorems 2.1 and 5.2 have nice features. The validity of theorem 2.1 is based essentially on the validity of the Peierls condition. It is evident from the proof of theorem 5.2 that the restriction to finite-range interactions is not necessary. It is also clear that it is the presence of stable droplets of the other phase, with arbitrary sizes, which prevents an analytic continuation of the pressure at the phase coexistence point, although the occurrence of large droplets is rare. This confirms therefore the arguments based on the droplet model. If $k \in \mathbb{N}$ is given (large enough), then a very large stable droplet of volume V, whose boundary is a contour Γ, contributes a factor

$$(\beta \Delta V)^k \mathrm{e}^{-\beta\|\Gamma\|}$$

to the k^{th}-derivative of the pressure. If the droplet has the largest volume, given its surface energy $\|\Gamma\|$, then V is of the order of $\|\Gamma\|^{\frac{d}{d-1}}$. Since $x^{k\frac{d}{d-1}}\mathrm{e}^{-x}$ has its maximum at $x = k\frac{d}{d-1}$, if

$$k = \left\lfloor \frac{d-1}{d}\beta\|\Gamma\| \right\rfloor ,$$

then this contribution is of the order of

$$C^k k!^{\frac{d}{d-1}} .$$

If very large contours are suppressed, then there is an analytic continuation of the pressure. This would also prevent the phenomenon of phase separation. This is precisely this phenomenon which is absent in a mean-field theory, in which equilibrium states are pure homogeneous states. It is true that one can consider in such theory interfaces and surface phenomena [vdW2] and [CHi]. However, one uses the *unstable* part of the isotherms, which is even more difficult to justify than the metastable part (see also below). Absence of analytic continuation and phase separation are linked together. Is it possible to have a proof of the absence of analytic continuation, which is based directly on the existence of the phenomenon of phase separation? Does the absence of an analytic continuation imply phase separation (perhaps in some weak sense only)?

An important question which remains unanswered is whether there is a possibility of an analytic continuation across the line of phase coexistence[51] in the complex plane, which is defined in proposition 3.1 by

$$\{z \in \mathbb{C} : \; z = \mu^*(\nu;\beta) + i\nu \;\forall\, \nu \in \mathbb{R}\} .$$

Most of the analysis presented here can be carried out, but the point where the proof fails is that *the contributions of large and thin contours are not anymore of the same sign.* In [F] Fisher proved that a droplet model may have such an analytic continuation. Langer in [La1] wrote a detailed paper on the analytic continuation inside the complex plane. However, one should be aware that existence or not of an analytic continuation is a very delicate question, as I have already shown in this paper. It is possible to define another droplet model[52] , where *no* analytic

[51] Here phase coexistence is defined by the fact that *all* contours are stable.

[52] J. Bricmont informed me about this model privately in 1993.

continuation is possible. Consider the function f defined on $\{h \in \mathbb{C} : \mathrm{Re}h \geq 0\}$ by the series

$$f(z) := \sum_{n=0}^{\infty} \exp\big(-\lambda n^{d-1} - h n^d\big) \quad \text{where } \lambda > 0 \text{ and } d \geq 2 \text{ (dimension).}$$

The droplets are here cubic droplets of linear sizes n. This series is a so-called lacunary series[53], i.e. of the form

$$\sum_{m=1}^{\infty} a_m z^{k(m)} \quad \text{where} \quad z = \mathrm{e}^{-h}\,,$$

with

$$k(m+1) - k(m) = (m+1)^d - m^d \to \infty \quad \text{if} \quad m \to \infty\,.$$

This series, as a consequence of a general theorem of complex analysis[54], Fabry's theorem [Rem2], has $\{z \in \mathbb{C} : |z| = 1\}$ has its natural boundary. Hence f, as a function of h, cannot be analytically continued from $\{h \in \mathbb{C} : \mathrm{Re}h > 0\}$ across $\mathrm{Re}h = 0$. For a brief discussion of these questions see [P] p.274. Penrose pointed out rightly in [P] that *Langer's derivation, however, uses the approximation of replacing an infinite series formula for the free energy of an Ising ferromagnet by the corresponding integral. Since analytic continuation is a form of extrapolation, the uncontrolled errors introduced by this approximation might have a profound effect on the analytically continued free energy.*

I have shown in these lectures that one cannot obtain analytic continuation of the isotherms at low temperatures for a large class of lattice models. Thus, the justification of the metastable part[55] of the isotherms, as analytic continuation of the equilibrium parts, which is often invoked, is *wrong*. On the other hand theorem 5.2 and its corollary show the role of the range of the interaction, and how analytic continuation is restored in the van der Waals limit. Theorem 5.3 and the discussion following it show that, if one is interested in the pressure and few derivatives of the pressure only, then one can neglect, from a computational viewpoint, the part sing^+_γ, provided γ is small. The pressure for the restricted phase has an analytic continuation, with an interaction of finite range γ^{-1}, for all h such that $\mathrm{Re}h \geq -1/8$, repectively all h such that $\mathrm{Re}h \leq 1/8$. This is in agreement with ideas put forward by

[53]A series $\sum_n a_n z^{\lambda_n}$ is *lacunary* if $\lim_n n/\lambda_n = 0$.

[54]For example, if one modifies the geometric series $\sum_n z^n$, by replacing it by the series

$$\sum_{n=0}^{m} z^n + \sum_{k>m} z^{k^2} \quad (m \text{ arbitrary large})$$

then this series has the boundary of the unit disc as natural boundary.

[55]Van Hove's paper [vH] excludes the possibility of justifying metastable isotherms in the thermodynamic limit using the basic principles of statistical mechanics, since the thermodynamic potentials are convex. In [LanRu] Lanford and Ruelle give a similar result at the level of states of the system. They show that for short range interactions the probability measures which are translation invariant solutions of the DLR equations are exactly the translation invariant states defined by the variational principle. *Consequently no solution of the DLR equation can be interpreted as describing a metastable state.* Assume that the pressure is not analytic at some activity z_0. Then the correlation functions of no state can be analytic in a neighbourhood U of z_0 and in the same time be solution of the DLR equations at some point of U. This result, however, does not give any information about a possibility or impossibility of an analytic continuation of the pressure from $z < z_0$ to $z > z_0$.

Penrose and Lebowitz in their seminal paper about metastability [PLeb]. Notice that one obtains only a metastable part for the pressure, since the restricted pressure $p^{+}_{r,\gamma}$ is convex. If part of the following quotation of Lebowitz [Leb] has been clarified, the principal problem, how to define metastable states precisely, with some justification from first principles, is not yet completely solved. From [Leb]: *the whole problem of metastable states represents somewhat of an embarrassment to rigorous statistical mechanics at the present time. For while the Van der Waals-Maxwell theory suggests that these states are the "analytic continuations" of the equilibrium state there are many who argue, Langer and Fisher among them, that this is one of the qualitative features of the infinite-range potential limit which does not persist for finite-range potentials. It is argued that in first-order phase transitions in real systems there is an essential singularity blocking analytic continuation. Even if this argument should turn out to be incorrect the question still remains of how to define (with or without analytic continuation) metastable states precisely, with some justification from first principles.* As pointed out by Lebowitz a theory of metastability should describe the familiar experimental facts about the large variety of metastable states occurring in nature[56]. A complete theory of metastability must then describe both the *static* properties of these states (there are metastable substances which are stable for millions of years), as well as the *dynamics* of their persistence and decay. A recent important work about rigorous treatment of metastability is [ScSh]; see also the forthcoming book of Olivieri and Vares [OV].

For surface or interfacial phenomena, at equilibrium, one should in principle base the whole theory on the partition function alone. This can be done in special cases. For example, Abraham and Reed computed the magnetization profile for a two-dimensional Ising model (see [AbRe]). Detailed study of this profile and of the intrinsic thickness of the interface is also possible (at low temperatures) with methods similar to those developed in these lectures [BrLebPf]. See [PfV] for another example of a surface phenomenon, the wetting of a wall. Nevertheless, such approaches are often difficult to implement, and therefore one needs approximate theories in which explicit results can be obtained. In this respect it would be useful to clarify the status of Cahn-Hilliard theory [CHi], which has been initiated by van der Waals [vdW2]. This type of theory is widely used today and is a successful approach. In most cases, the free energy functional, which is non-convex below

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[56] One should also not forget that the notion of "equilibrium" is a theoretical notion. The following quotation from Callen's book on Thermodynamics is pertinent. From [Ca] p.15: *In actuality, few systems are in absolute and true equilibrium. In absolute equilibrium all radioactive materials would have decayed completely and nuclear reactions would have transmuted all nuclei to the most stable of isotopes. Such processes, which would take cosmic times to complete, generally can be ignored. A system that has completed the relevant processes of spontaneous evolution, and that can be described by a reasonably small number of parameters, can be considered to be in* metastable equilibrium. *Such a limited equilibrium is sufficient for the application of thermodynamics.*
In practice the criterion for equilibrium is circular. Operationally, a system is in equilibrium state if its properties are consistently described by thermodynamics theory! *It is important to reflect upon the fact that the circular character of thermodynamics is* not *fundamentally different from that of mechanics.*

In [Fe], p.1, Feynman defines the notion of thermal equilibrium as follows: *If a system is very weakly coupled to a heat bath at a given "temperature," if the coupling is indefinite or not known precisely, if the coupling has been on for a long time, and if all the "fast" things have happened and all the "slow" things not, the system is said to be in* thermal equilibrium.

the critical temperature, is treated as a given phenomenological quantity. Is Cahn-Hilliard theory only a mean-field type theory? Can it be derived as approximate theory in some *controlled way* for some systems with finite-range interactions? The justification of these non-convex functionals in Cahn-Hilliard theory is discussed by Langer in [La2]. The argumentation does not differ very much from the ideas of van Kampen [vK]. It is a coarse-grained approach. One point is worth mentioning in Langer's argumentation about the size for the coarse-graining cells: they should be neither too small, nor too large so that phase separation cannot occur within single cells. For more details the reader is referred to [La2].

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