KAC SEMINAR

UTRECHT, FEBRUARY 2006

RANDOM MOTIONS IN RANDOM MEDIA (I)

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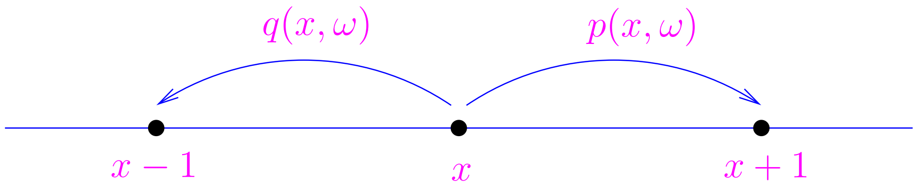
RANDOM MEDIA APPEAR IN SEVERAL WAYS.

TWO NATURAL MODELS:

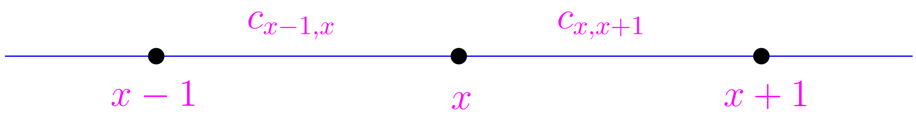
- SITE RANDOMNESS: $p(x,\omega)$ i.i.d.



Chernov (67), Temkin (72)

RANDOM WALK IN RANDOM ENVIRONMENT

- BOND RANDOMNESS: $c_{x,x+1}(\omega) > 0$ i.i.d.



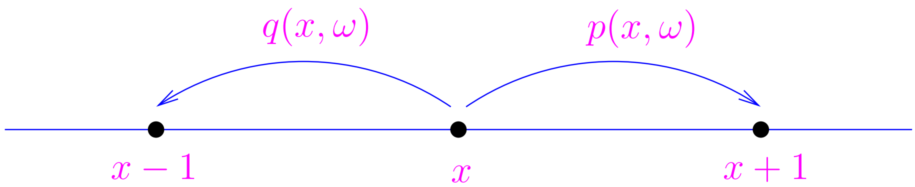
$$p(x,\omega) = \frac{c_{x,x+1}(\omega)}{c_{x-1,x}(\omega) + c_{x,x+1}(\omega)}$$

Fatt (56), Kirkpatrick (71)

RANDOM CONDUCTANCE MODEL

BOTH MODELS EASILY GENERALIZED TO $d > 1$.

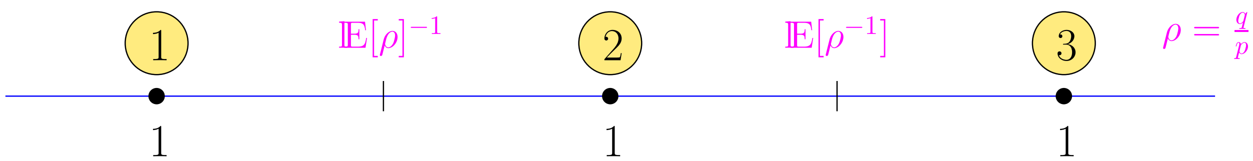
RANDOM WALK IN RANDOM ENVIRONMENT (RWRE)



FIRST NAIVE GUESS: $\dfrac{X_n}{n} \longrightarrow \mathbb{E}[p] - \mathbb{E}[q]$

WRONG!

IN FACT WHAT HAPPENS: *Solomon (75)*



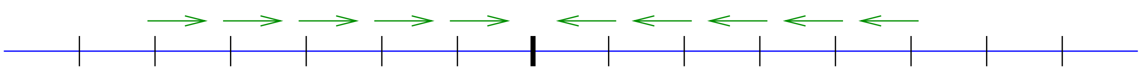
$\dfrac{X_n}{n} \longrightarrow v$ WHERE

1 $\quad 0 < v = \dfrac{1-\mathbb{E}[\rho]}{1+\mathbb{E}[\rho]} < \mathbb{E}[p] - \mathbb{E}[q]$

2 $\quad v = 0$

3 $\quad 0 > v = \dfrac{\mathbb{E}[\rho^{-1}]-1}{\mathbb{E}[\rho^{-1}]+1} > \mathbb{E}[p] - \mathbb{E}[q]$

SLOWDOWN EFFECTS ! TRAPS

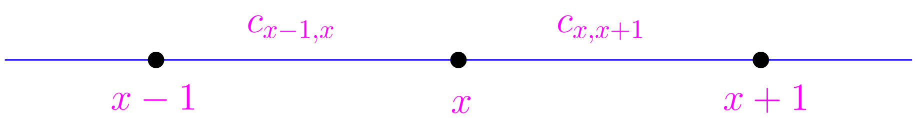


IN FACT *Sinai (82)*

WHEN $\mathbb{E}[\log\rho] = 0$, $0 < \mathbb{E}[(\log\rho)^2] < \infty$ (CASE 2)

$$X_n \sim (\log n)^2 .$$

RANDOM CONDUCTANCE MODEL (RCM)



$$\frac{X_n}{n} \longrightarrow v = 0, \qquad \frac{X_{[\cdot n]}}{\sqrt{n}} \longrightarrow BM(\sigma^2)$$

ENVIRONMENT VIEWED FROM PARTICLE:

Kozlov, Papanicolaou-Varadhan (79)

$\overline{\omega}_n = \omega(X_n + \cdot)$

MARKOV CHAIN (HUGE STATE SPACE)

LOOK FOR INVARIANT MEASURE

$$\underbrace{\mathbb{Q}}_{\text{dynamic}} = f \, \underbrace{\mathbb{P}}_{\text{static}}$$

"COMPARABLE POINTS OF VIEWS"

FOR RCM:

$$\mathbb{Q} = \frac{1}{Z}\left(c_{-1,0}(\omega) + c_{0,1}(\omega)\right)\mathbb{P}$$

REVERSIBLE MEASURE

CLT FOR $d > 1$, ERGODIC ENVIRONMENTS

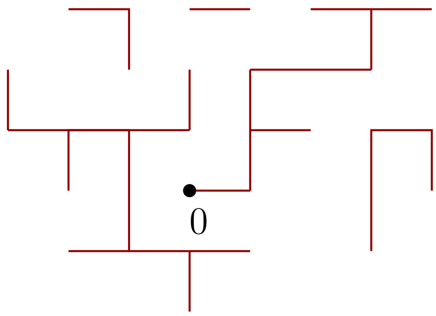
Kunnemann (83), Kozlov (85), Kipnis-Varadhan (86), De Masi et al. (89), ...

"SPECIAL CASE":

R.W. ON PERCOLATION CLUSTERS

"ANT IN THE LABYRINTH" *De Gennes (76)*

$d \geq 2$, $c_{x,x+e} = 1$ prob α, $= 0$ prob $1-\alpha$



1 $\alpha > \alpha_c(d)$ UNIQUE INFINITE CLUSTER $\mathcal{C}(\omega)$

2 $\alpha < \alpha_c(d)$ ONLY FINITE COMPONENTS

FOR 1 $0 \in \mathcal{C}(\omega)$ $\quad \dfrac{X_{[\cdot n]}}{\sqrt{n}} \longrightarrow BM(\sigma^2)$

De Masi et al. (86), Sidoravicius-Sznitman (04)
Berger-Biskup (05), Mathieu-Piatnitski (05)

CASE WITH ANISOTROPY: *Barma-Dhar (83), ...*

$$p_\omega(x,y) = \frac{e^{\ell\cdot(y-x)} c_{x,y}(\omega)}{\sum\limits_{z\sim x} e^{\ell\cdot(z-x)} c_{x,z(\omega)}}\,, \quad y \sim x$$

$\ell \neq 0$ "ANISOTROPY"

IN DETERMINISTIC MEDIUM $c_{x,y} \equiv 1$, $x \sim y$

$$\frac{X_n}{n} \longrightarrow v = \sum_{i=1}^{d} \frac{\sinh(\ell\cdot e_i)\, e_i}{\sum\limits_j \cosh(\ell\cdot e_j)} \neq 0\,.$$

IN PERCOLATION CASE

- FOR ALL $\ell \neq 0$

$$X_n \cdot \ell \longrightarrow \infty$$

TRANSIENCE IN DIRECTION ℓ

- WEAK ANISOTROPY: $0 < |\ell| < \lambda_0(d,\alpha)$

$$\frac{X_n}{n} \longrightarrow v \neq 0, \quad \frac{X_{[\cdot n]} - [\cdot n]v}{\sqrt{n}} \longrightarrow BM(A)$$

BALLISTIC BEHAVIOR

- STRONG ANISOTROPY: $\lambda_1(d,\alpha) < |\ell|$

$$\frac{|X_n|}{n^{\lambda_1/|\ell|}} \longrightarrow 0$$

SUB-BALLISTIC, EVEN SUB-DIFFUSIVE BEHAVIOR

Berger-Gantert-Peres (03), Sznitman (03)

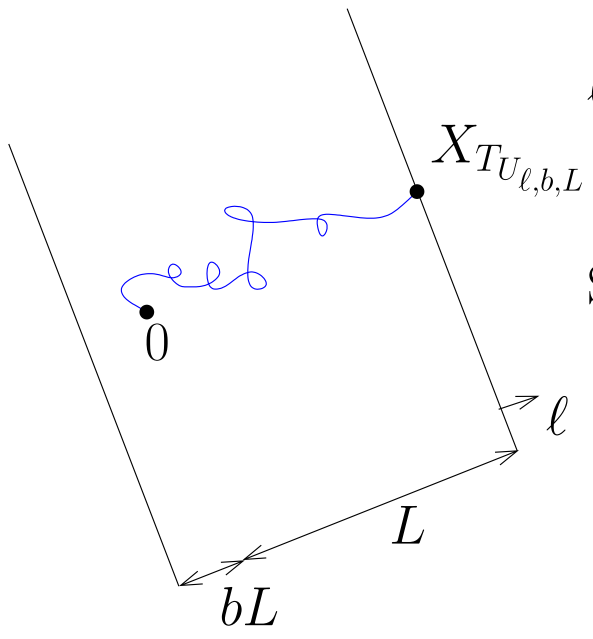
BACK TO RWRE

- $d = 1$ EXTENSIVELY STUDIED
  Temkin (72), Solomon (75),
  Kesten-Kozlov-Spitzer (75), Sinai (82), ...

- $d > 1$ MATHEMATICAL CHALLENGE:
  GENUINE NON-REVERSIBLE
  Kalikow (81), Lawler (82), Bricmont-Kupiainen (91),

  RECENTLY MUCH WORK
  PROGRESS ON BALLISTIC WALKS

CONDITIONS (T) AND (T')



$\ell \in S^{d-1}, b, L > 0$

Slab $U_{\ell,b,L}$

DEF. $0 < \gamma \leq 1$, $\ell \in S^{d-1}$

$(T_\gamma)|\ell \iff$ for all ℓ' in a neighb. of ℓ

$$\overline{\lim_{L\to\infty}}\, L^{-\gamma} \log P_0\big[X_{T_{U_{\ell',b,L}}} \cdot \ell' < 0\big] < 0, \ \forall b > 0$$

$$(T)|\ell \equiv (T_{\gamma=1})|\ell, \quad (T')|\ell \equiv (T_\gamma)|\ell \quad \text{for all } 0 < \gamma < 1,$$

$$\text{so} \quad (T)|\ell \Longrightarrow (T')|\ell \Longrightarrow (T_\gamma)|\ell, \quad 0 < \gamma < 1.$$

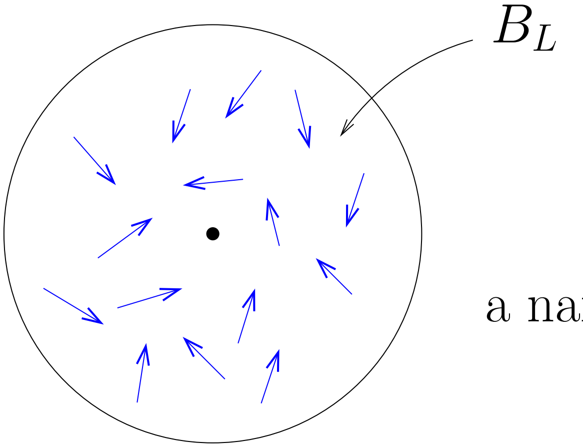
$d \geq 2$ IMPORTANT CONSEQUENCES:

THEOR.: ($d \geq 2, (T')|\ell$, *Sznitman (99),(00)*)

- LLN P_0-A.S. $\dfrac{X_n}{n} \longrightarrow v$ determ., $v \cdot \ell > 0$
- CLT $B^n_\cdot = \dfrac{1}{\sqrt{n}}\left(X_{[\cdot n]} - [\cdot n]v\right) \xrightarrow[P_0]{\text{LAW}} \underset{\substack{\uparrow \\ \text{NON}-\text{DEG.}}}{BM(A)}$
- SLOWDOWNS (CRIT. L.D.)

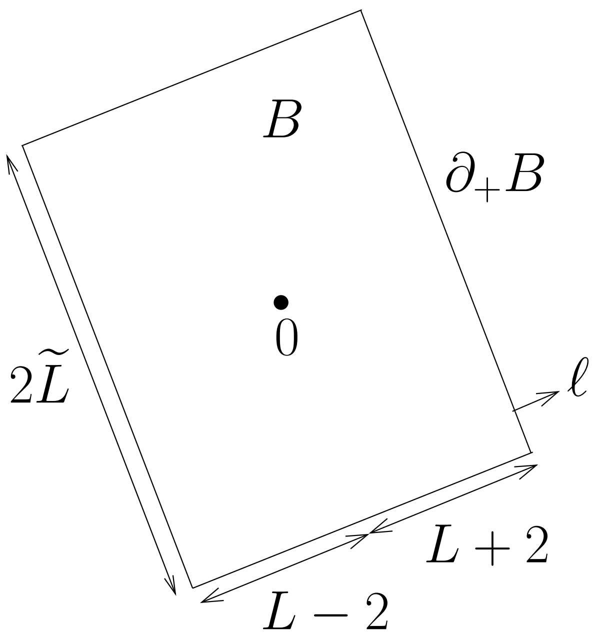
PROOF INVOLVES REGENERATION TIME τ_1
(RENEWAL STRUCTURE) AND KEY ESTIMATE

$$P_0[\tau_1 > n] \leq e^{-K(\log n)^\alpha}, \quad \alpha < \frac{2d}{d+1}\,.$$



a naive trap

EFFECTIVE CRITERION



$$\rho_B = \frac{P_{0,\omega}[X_{T_B} \notin \partial_+ B]}{P_{0,\omega}[X_{T_B} \in \partial_+ B]}$$

THEOR.: *(Sznitman (02))*

$$(T')|\ell \iff \inf_{B, 0<\alpha\leq 1} \{A\, \mathbb{E}[\rho_B^\alpha]\} < 1$$

$$L \geq c_2(d),\ \widetilde{L} \in (3\sqrt{d}, L^3)$$

$$A = c_1(d)\Big(\log\frac{1}{\kappa}\Big)^{3(d-1)} \widetilde{L}^{(d-1)}\, L^{3(d-1)+1}$$

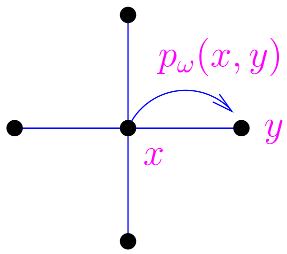
ellipticity

WITH EFF. CRITERION:

- $(T_\gamma)|\ell \iff (T')|\ell$, WHEN $\frac{1}{2} < \gamma < 1$.
- $(T')|\ell$ STABLE UNDER PERTURB.
- CONSTRUCT NEW EXAMPLES:

$$d(x,\omega) = \sum_{y \sim x} (y-x)\, p_\omega(x,y)$$

LOCAL DRIFT AT x



ε-PERTURB. OF S.R.W.: $\left| p_\omega(x,y) - \frac{1}{2d} \right| \le \varepsilon$

THEOR.: *($d \ge 3$, Sznitman (03))*

$\eta > 0$, FOR $\varepsilon < \varepsilon_0(d,\eta)$ and ε - pert. of S.R.W.,
when $\mathbb{E}[d(0,\omega)\cdot e_1] > \varepsilon^{\frac{5}{2}-\eta}$, $d=3$, $> \varepsilon^{3-\eta}$, $d \ge 4$,

$$\frac{X_n}{n} \longrightarrow v \text{ with } v\cdot e_1 > 0, \quad \frac{X_{[\cdot n]} - [\cdot n]v}{\sqrt{n}} \longrightarrow BM(A)\,.$$

GIVES NEW EXAMPLES OF BALLISTIC BEH. WITH

$$d(0,\omega) \sim \varepsilon, \text{ BUT } \mathbb{E}[d(0,\omega)] \ll \varepsilon^2$$

beyond Kalikow's condition!

CAN ONE SIMPLY ASSUME $\mathbb{E}[d(0,\omega)\cdot e_1] > 0$?

THEOR.: *(Bolthausen-Sznitman-Zeitouni (03))*

$d \geq 7$, FOR EACH ε, THERE ARE ε-PERTURB. OF S.R.W. WITH $\mathbb{E}[d(0,\omega)\cdot e_1] > 0$, BUT

$$\frac{X_n}{n} \longrightarrow 0\,.$$

IF $d \geq 15$, EVEN $\dfrac{X_{[\cdot n]}}{\sqrt{n}} \longrightarrow BM(A)$

ONE CAN ALSO CONSTRUCT EXAMPLES

$$\frac{X_n}{n} \longrightarrow v \neq 0 \text{ with } v = -c\,\mathbb{E}[d(0,\omega)].$$

(IMPOSSIBLE WHEN $d = 1$)

WITH ERGODIC ENV. EXAMPLE

(Zerner-Merkl (01), Bramson-Zeitouni-Zerner (05))

$$P\Big[\frac{X_n}{n} \to v\Big] = \frac{1}{2} = P\Big[\frac{X_n}{n} \to -v\Big]$$

SOME RELATED SURVEYS:

E. Bolthausen and A.-S. Sznitman: "Ten lectures on random media", DMV-Lectures, vol. 32, Birkhäuser, (2002)

A.-S. Sznitman: "Milieux alatoires et petites valeurs propres", Panorama et Synthses, SMF, 12, 13-36, (2001)

A.-S. Sznitman: "Topics in random walks in random environment", ICTP Lecture Notes Series, vol.17, Trieste, 203-266, (2004), also available at http://www.ictp.trieste.it/∼pub_off/lectures/

A.-S. Sznitman: "Random motions in random media", mini-course Les Houches Summer School, also available at http://www.math.ethz.ch/u/sznitman/preprints

O. Zeitouni: "Random walks in random environment", Lecture Notes in Math. 1837, 190-312, (2004)