Deviation inequalities: an overview

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Denote

$$\delta_i f = \sup_\sigma f(\sigma^i) - f(\sigma)$$

the maximal effect of a spin-flip at site i for the function f, and, for $p \geq 1$

$$\|\delta f\|_p^p := \sum_i (\delta_i f)^p$$

for $p = 1$ this is the "triple norm" used by Liggett.

1 Gaussian and moment deviation bounds

1. **Gaussian deviation bound**

$$\mathbb{P}_\mu\left(|f - \mathbb{E}_\mu(f)| > t\right) \leq 2e^{-c\frac{t^2}{\|\delta f\|_2^2}} \tag{1.1}$$

   is equivalent with **the exponential moment bound**

$$\mathbb{E}_\mu(e^{\lambda(f-\mathbb{E}_\mu(f))}) \leq e^{c'\lambda^2\|\delta f\|_2^2} \tag{1.2}$$

2. **Devroye inequality**

$$Var_\mu(f) = \mathbb{E}_\mu(f - \mathbb{E}_\mu(f))^2 \leq C\|\delta f\|_2^2 \tag{1.3}$$

3. **Higher moment bounds**

$$\mathbb{E}_\mu(f - \mathbb{E}_\mu(f))^{2p} \leq C_p\|\delta f\|_2^{2p} \tag{1.4}$$

   for $p > 1$.

2 Log-Sobolev and Poincare inequalities

Denote

$$\mathcal{E}(f,f) = \sum_i \int (\nabla_i f)^2 d\mu$$

this can be replaced by a more general Dirichlet form. In our context

$$\nabla_i f(\sigma) = f(\sigma^i) - f(\sigma)$$

Associated to a measure μ, we consider the reversible Glauber dynamics with generator

$$Lf = \sum_i c_i \nabla_i f \tag{2.1}$$

with c_i the flip rates such that

$$\mu(\sigma)c_i(\sigma) = \mu(\sigma^i)c_i(\sigma^i)$$

making μ into a reversible measure. The semigroup of the process generated by L is denoted by S_t.

1. **Log Sobolev inequality**

$$Ent_\mu(f^2) := \mathbb{E}_\mu\left(f^2 \log\left(\frac{f^2}{\mathbb{E}_\mu(f^2)}\right)\right) \le c\mathcal{E}(f,f) \tag{2.2}$$

2. **Poincaré inequality**

$$Var_\mu(f) \le c\mathcal{E}(f,f) \tag{2.3}$$

Log Sobolev implies Poincaré. Poincare implies exponential relaxation in $L^2(\mu)$, i.e.,

$$Var_\mu(S_t f) \le e^{-ct}\|f\|_2^2$$

Log-Sobolev implies exponential relaxation in $L^\infty(\mu)$.

3. **Weak log-Sobolev inequality**

There exist $s \mapsto \beta(s)$ such that for all $s > 0$

$$Ent_\mu(f^2) \le \beta(s)\mathcal{E}(f,f) + s\|\delta f\|_2^2 \tag{2.4}$$

4. **Weak Poincaré inequality**

There exist $s \mapsto \beta(s)$ such that for all $s > 0$

$$Var_\mu(f) \le \beta(s)\mathcal{E}(f,f) + s\|\delta f\|_2^2 \tag{2.5}$$

Typically, $\beta(s) \to \infty$ as $s \to 0$, otherwise we are back in the original case. Weak Poincaré implies relaxation of the semigroup with an estimate of the form

$$E_\mu((S_t f)^2) \le \xi(t)\|\delta f\|_2^2$$

for f with $\int f d\mu = 0$, with ξ_t related to β:

$$\xi_t = \inf\{r > 0 : -\beta(r)\log(r) \leq 2t\}$$

idem for weak log-Sobolev and $L^\infty(\mu)$ relaxation. Weak log-Sobolev implies weak Poincare with

$$\beta_{WP}(s) = \frac{24\beta_{WL}(s/2\log(1+s/2))}{\log(1+(1/2s))}$$

Weak log-Sobolev implies *ordinary* Poincare if

$$\beta_{WP}(s) \leq c_1 \log(c_2/s)$$

for s small enough.

5. **Log-Sobolev version Bobkov-Götze**

$$Ent_\mu(e^f) \leq c \sum_i \int e^f (\nabla_i(f))^2 d\mu \tag{2.6}$$

Connections with previous inequalities:

a) Poincare implies Devroye.

b) Log Sobolev implies Gaussian deviation bound (Herbst argument) in the case the Dirichlet form comes from a *derivation*, e.g., for measures on $\mathbb{R}^n$, and

$$\mathcal{E}(f,f) = \int (\nabla f)^2 dx$$

with ∇ the (ordinary) gradient.

c) Log-Sobolev version Bobkov-Götze implies Gaussian deviation bound.

3 Transportation cost inequalities

For two probability measures μ, ν on a metric space, we define the Wasserstein distances:

$$W_p(\mu,\nu) = \inf\{\left(\int d(x,y)^p \mathbb{P}(dxdy)\right)^{1/p} : \mathbb{P}_1 = \mu, \mathbb{P}_2 = \nu\} \tag{3.1}$$

only $p = 1, 2$ are of interest to us here. The information "distance" or relative entropy "distance" is defined via

$$h(\nu|\mu) = \int \log\left(\frac{d\nu}{d\mu}\right) d\nu \tag{3.2}$$

A transportation cost inequality $TC(p)$ is an inequality of the form

$$W_p(\nu,\mu) \leq \sqrt{2Ch(\nu|\mu)}$$

Implications with previous inequalities:

a) TC (1) is equivalent with Gaussian deviation bound

b) Log-Sobolev inequality implies TC(2).

References

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