Brownian motion, Ricci curvature, functional inequalities and geometric flows

Anton Thalmaier
University of Luxembourg

Mark Kac Seminar
Utrecht, Netherlands

March 2, 2018

Outline

1. Stochastic analysis of static and evolving manifolds
2. Characterizing Ricci curvature by functional inequalities
3. Heat equations under geometric flows and entropy formulas

I. Motivation: Heat equation on a Riemannian manifold

- Let (M,g) be a complete Riemannian manifold (M,g) and

$$L = \Delta + Z \quad \text{with } Z \in \Gamma(TM)$$

- u be a positive solution to

$$\frac{\partial}{\partial t} u = Lu \quad \text{on} \quad M \times \mathbb{R}_+$$

- (Gradient estimate) Want to bound

$$|\nabla u| \text{ or } \frac{|\nabla u|}{u}.$$

- (Harnack inequalities) Want to compare

$$u(x,s) \text{ and } u(y,t).$$

- Why is Ricci curvature important for such questions?

- **Cheng-Yau** (1975)
  Let M be complete and D be some open, relatively compact domain D in M. Assume that u is a positive harmonic function on D:

$$\Delta u = 0$$

Then

$$\frac{|\nabla u|}{u}(x) \leq c(n)\left[\sqrt{K} + \frac{1}{r(x)}\right]$$

if $\mathrm{Ric}|D \geq -K$, $K \geq 0$ (where $r(x) = \mathrm{dist}(x, \partial D)$ and $n = \dim M$).

The formula is easy to prove by probabilistic methods, e.g. Arnaudon, Driver, Th. (2007).

- For $L = \Delta + Z$ let u be a solution to $\frac{\partial}{\partial t} u = Lu.$

  There is an exact formula for the differential

$$(\nabla u)(\cdot, t)_x$$

  in terms of an L-diffusion starting from x:

$$X_t = X_t^x, \quad t < \zeta(x).$$

- Recall that L-diffusions X_t on M are defined by the property that for each $f \in C_c^\infty(M)$,

$$d(f(X_t)) - (Lf)(X_t)\, dt = 0$$

  (mod differentials of loc mart.)

- Denote by

$$\mathrm{Ric}^Z = \mathrm{Ric} - \nabla Z$$

the Bakry-Émery Ricci tensor, i.e.

$$\mathrm{Ric}^Z(X, Y) := \mathrm{Ric}(X, Y) - \langle \nabla_X Z, Y \rangle .$$

- Let

$$\mathrm{Ric}^Z_{//_t} := //_t^{-1} \circ \mathrm{Ric}^Z_{X_t} \circ //_t \in \mathrm{End}(T_x M)$$

where $//_t \colon T_x M \to T_{X_t} M$ is parallel transport along $X_t = X_t^x$:

$$\begin{array}{ccc} T_x M & \overset{\mathrm{Ric}^Z_{//_t}}{\dashrightarrow} & T_x M \\ {\scriptstyle //_t}\downarrow & & \uparrow{\scriptstyle //_t^{-1}} \\ T_{X_t} M & \xrightarrow[\mathrm{Ric}^Z_{X_t}]{} & T_{X_t} M \end{array}$$

By convention $\mathrm{Ric}^Z_x(v) = \mathrm{Ric}^Z_x(\cdot, v)^\sharp$ for $v \in T_x M$.

Damped parallel transport

- For $x \in M$ define a linear transformation

$$Q_t \colon T_x M \to T_x M$$

as solution to the pathwise ODE

$$\begin{cases} dQ_t = -Q_t \, \mathrm{Ric}^Z_{//_t} \, dt \\ Q_0 = \mathrm{id}_{T_x M} \end{cases}$$

- In the sequel we need

$$Q_t \circ //_t^{-1} \colon T_{X_t} M \to T_x M$$

("damped parallel transport" along X_t)

Theorem (Gradient formulas)

Let $f \in \mathscr{B}_b(M)$ *and* $u(x,t) = P_t f(x)$ *be the (minimal) solution to*

$$\frac{\partial}{\partial t} u = Lu, \; u|_{t=0} = f.$$

- *(Semigroup formula) Then* $P_t f(x) = \mathbb{E}[f(X_t^x) 1_{\{t<\zeta(x)\}}]$.
- *(Derivative formula) If* $f \in C_b^1(M)$ *and* Ric^Z *bounded below,*

$$(\nabla P_t f)(x) = \mathbb{E}\left[Q_t //_t^{-1} \nabla f(X_t^x)\right]$$

- *(Bismut formula) If* $f \in \mathscr{B}_b(M)$ *(no assumption on* Ric*), then*

$$\langle (\nabla P_t f)_x, v \rangle = -\mathbb{E}\left[f(X_t^x) 1_{\{t<\zeta(x)\}} \int_0^\tau \left\langle Q_s^* \dot{\ell}_s, dB_s \right\rangle \right]$$

  for each $v \in T_x M$*, where*
  - $\tau = \tau_D(x) \wedge t$ *with* $\tau_D(x)$ *the first exit time of* X_t^x *from some relatively compact neighbourhood* D *of* x
  - B *is a Brownian motion in* $T_x M$
  - ℓ_t *is any adapted process in* $T_x M$ *with absolutely continuous paths of finite energy such that* $\ell_0 = v$ *and* $\ell_\tau = 0$.

A first observation

- Suppose that

$$\mathrm{CD}(K,\infty) \qquad \mathrm{Ric}^Z(X,X) \geq K|X|^2, \quad X \in TM,$$

  for some constant K.

- Then

$$|Q_t| \leq e^{-Kt}, \quad t \geq 0.$$

- Hence,

$$\text{(gradient estimate)} \qquad |\nabla P_t f| \leq e^{-Kt} P_t |\nabla f|^2, \quad f \in C_b^1(M).$$

- Actually the gradient estimate is equivalent to $\mathrm{CD}(K,\infty)$.

II. Stochastic flows

Let L be a second order PDO on M, e.g.

$$L = A_0 + \sum_{i=1}^{r} A_i^2,$$

where $A_0, A_1, \ldots, A_r \in \Gamma(TM)$ for some $r \in \mathbb{N}$.

Let

$$X_{\bullet}^x \equiv (X_t^x)_{t\geq 0}$$

be an M-valued *L-diffusion* (or *flow process* to L) with starting point x in the sense that $X_0^x = x$ and for all $f \in C_c^\infty(M)$, the process

$$N_t^f(x) := f(X_t^x) - f(x) - \int_0^t (Lf)(X_s^x)\, ds, \quad t \geq 0,$$

is a martingale, i.e.

$$\mathbb{E}^{\mathscr{F}_s} \underbrace{\left[f(X_t^x) - f(X_s^x) - \int_s^t (Lf)(X_r^x)\, dr \right]}_{= N_t^f(x) - N_s^f(x)} = 0, \quad \text{for all } s \leq t.$$

Recall Let Z be a Brownian motion on $\mathbb{R}^r$. Then solutions X to the Stratonovich SDE on M:

$$\boxed{dX = A_0(X)\,dt + \sum_{i=1}^{r} A_i(X) \circ dZ^i}$$

are L-diffusions to the operator

$$L = A_0 + \sum_{i=1}^{r} A_i^2$$

Brownian motions and moving frames

Brownian motions on M are L-diffusions (stochastic flows) to the Laplace-Beltrami operator Δ on M.

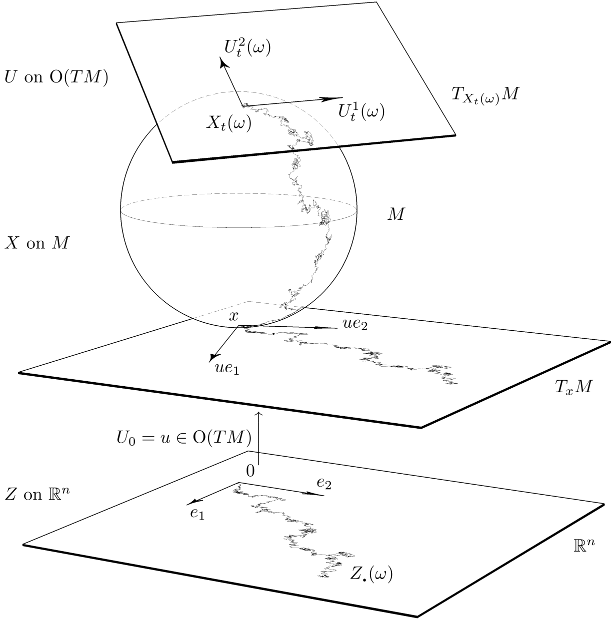
Good: We have a method to construct Brownian motions.

Bad: There is no canonical way to write Δ in Hörmander form as a sum of squares.

Notation. Let $\pi\colon P \to M$ be the G-principal bundle of orthonormal frames with $G = \mathrm{O}(n)$. The fibre P_x consists of the linear isometries $u\colon \mathbb{R}^n \to T_xM$ where $u \in P_x$ is identified with the $\mathbb{R}$-basis

$$(u_1, \ldots, u_n) := (ue_1, \ldots, ue_n).$$

Write $P = \mathrm{O}(TM)$.


U on $\mathrm{O}(TM)$
$U_t^2(\omega)$
$U_t^1(\omega)$
$X_t(\omega)$
$T_{X_t(\omega)}M$
M
X on M
x
ue_2
ue_1
T_xM
$U_0 = u \in \mathrm{O}(TM)$
0
Z on $\mathbb{R}^n$
e_2
e_1
$\mathbb{R}^n$
$Z_\bullet(\omega)$

The Levi-Civita connection in TM induces canonically a *G-connection* in P given as a G-invariant differentiable splitting h of the following exact sequence of vector bundles over P:

$$0 \longrightarrow \ker d\pi \longrightarrow TP \underset{\underset{h}{\curvearrowleft}}{\xrightarrow{d\pi}} \pi^* TM \longrightarrow 0.$$

The splitting gives a decomposition of TP:

$$TP = V \oplus H := \ker d\pi \oplus h(\pi^* TM).$$

For $u \in P$, the space H_u is called the *horizontal space at u* and $V_u = \{v \in T_u P : \ (d\pi)v = 0\}$ the *vertical space at u*.

The bundle isomorphism

$$h : \ \pi^* TM \xrightarrow{\sim} H \hookrightarrow TP$$

is the *horizontal lift* of the G-connection; fibrewise it reads as

$$h_u : \ T_{\pi(u)} M \xrightarrow{\sim} H_u.$$

- The orthonormal frame bundle $P = \mathrm{O}(TM)$, considered as a manifold, is parallelizable.
- The horizontal subbundle H is trivialized by the *standard-horizontal vector fields* $H_1, \dots, H_n$ in $\Gamma(TP)$ defined by

$$H_i(u) := h_u(ue_i).$$

- The canonical second order partial differential operator on $\mathrm{O}(TM)$,

$$\Delta^{\mathrm{hor}} := \sum_{i=1}^{n} H_i^2,$$

is called Bochner's *horizontal Laplacian*.

(a) Let Z be a semimartingale on $\mathbb{R}^n$. Solve the following SDE on the frame bundle $P = \mathrm{O}(TM)$:

$$dU = \sum_{i=1}^{n} H_i(U) \circ dZ^i, \quad U_0 = u_0.$$

(b) Project U onto the manifold M:

$$X = \pi \circ U$$

(c) From X we can recover again Z via $Z = \int_U \vartheta$ where U is the unique horizontal lift of X to P with $U_0 = u_0$ and

$$\vartheta \in \Gamma(T^*P \otimes \mathbb{R}^n), \quad \vartheta_u(X_u) := u^{-1}(d\pi X_u), \quad u \in P,$$

the canonical 1-form.

We call X on M *stochastic development* of Z. The frame U moves along X by stochastic parallel transport.

Theorem (Stochastic development)

The following three conditions are equivalent:

- *Z is a Brownian motion on $\mathbb{R}^n$ (diffusion with generator $\Delta_{\mathbb{R}^n}$).*
- *U is an L-diffusion on $P = \mathrm{O}(TM)$ to*

$$L = \Delta^{\mathrm{hor}} = \sum_{i=1}^{n} H_i^2.$$

- *X is a Brownian motion on M (diffusion with generator the Laplace-Beltrami operator Δ on M).*

Indeed: Use that

$$\Delta^{\mathrm{hor}}(f \circ \pi) = (\Delta f) \circ \pi$$

Definition (Parallel transport along a semimartingale)

For $0 \le s \le t$, consider

$$\begin{array}{ccc} T_{X_s}M & \xrightarrow[\sim]{\hspace{3em}} & T_{X_t}M \\ & {\scriptstyle U_s}\nwarrow \quad \nearrow{\scriptstyle U_t} & \\ & \mathbb{R}^n & \end{array}$$

The isometries

$$//_{s,t} := U_t \circ U_s^{-1} : \; T_{X_s}M \to T_{X_t}M$$

are called *stochastic parallel transport along* X.

III. Derivative formulas

- (Process) X_t is an L-diffusion where

$$L = \Delta + Z \quad \text{with } Z \in \Gamma(TM)$$

- Let $\mathrm{Ric}^Z = \mathrm{Ric} - \nabla Z$, i.e.

$$\mathrm{Ric}^Z(X,Y) = \mathrm{Ric}(X,Y) - \langle \nabla_X Z, Y\rangle.$$

- Corresponding semigroup:

$$P_t f(x) = \mathbb{E}[f(X_t^x)\, 1\!\!1_{\{t<\zeta(x)\}}], \quad t \geq 0.$$

Goal: Stochastic formula for $\nabla P_t f$!

- **Basic observation**

  Let Q_t be the $\mathrm{Aut}(T_xM)$-valued process defined by

$$\frac{d}{dt}Q_t = -Q_t\,(\mathrm{Ric}_Z)_{//_t}, \quad Q_0 = \mathrm{id}_{T_xM}.$$

  Fix $t > 0$. Then,

$$N_s := Q_s //_s^{-1} (\nabla P_{t-s} f)_{X_s^x}, \quad 0 \le s \le t,$$

  is a local martingale in T_xM.

- **How to check?** Write everything as functions on $\mathrm{O}(TM)$, e.g. to $a \in \Gamma(TM)$ consider

$$F_a\colon\ \mathrm{O}(TM) \to \mathbb{R}^n, \quad F_a(u) = u^{-1} a_{\pi(u)}.$$

  Letting $a_t := \nabla P_t f$, we have

$$N_s = (Q_s U_0) \cdot F_{a_{t-s}}(U_s).$$

  Use Itô's formula to calculate dN_s.

- Suppose that

$$N_s = Q_s //_s^{-1} (\nabla P_{t-s} f)_{X_s^x}, \quad 0 \le s \le t,$$

  is a true martingale.
- Then the equality $\mathbb{E}[N_0] = \mathbb{E}[N_t]$ gives the following derivative formula

$$(\nabla P_t f)(x) = \mathbb{E}\Big[Q_t //_t^{-1} \nabla f(X_t^x)\Big], \quad t \ge 0.$$

- This formula clearly requires conditions on boundedness of Ric^Z from below. It can not hold in case of explosion of the $(\Delta + Z)$-diffusion.

Fix $t > 0$. Since $N_s = Q_s //_s^{-1} (\nabla P_{t-s} f)_{X_s^x}$ is a local martingale, for any adapted process ℓ_s with absolutely continuous paths,

$$n_s := \langle N_s, \ell_s \rangle - \int_0^s \langle N_r, d\ell_r \rangle$$
$$= \Big\langle (\nabla P_{t-s} f)_{X_s^x}, //_s \, Q_s^* \, \ell_s \Big\rangle - \int_0^s \Big\langle (\nabla P_{t-r} f)_{X_r^x}, //_r \, Q_r^* \, \dot{\ell}_r \Big\rangle \, dr$$

is a local martingale as well ($0 \le s \le t$). Thus

$$n_s' := \langle (\nabla P_{t-s} f)_{X_s^x}, //_s \, Q_s^* \, \ell_s \rangle - \int_0^s \Big\langle (\nabla P_{t-r} f)_{X_r^x}, //_r \, dB_r \Big\rangle \int_0^s \langle Q_s^* \, \dot{\ell}_r, dB_r \rangle$$

is a local martingale. But since

$$(P_{t-s} f)(X_s^x) = \int_0^s \langle (\nabla P_{t-r} f)_{X_r^x}, //_r \, dB_r \rangle,$$

we finally see that

$$\langle (\nabla P_{t-s} f)_{X_s^x}, //_s Q_s^* \, \ell_s \rangle - (P_{t-s} f)(X_s^x) \int_0^s \langle Q_r^* \, \dot{\ell}_r, dB_r \rangle, \quad 0 \le s \le t,$$

is a local martingale.

- Choose ℓ_s such that the local martingale n'_s is a true martingale, and further such that $\ell_0 = v$ and $\ell_t = 0$.
- This can always be achieved by taking $\ell_s = 0$ for $s \geq t \wedge \tau(x)$ where $\tau(x)$ is the first exit time of X_s^x from a relatively compact neighborhood of x.
- The equality

$$\mathbb{E}[n'_0] = \mathbb{E}[n'_{t\wedge\tau(x)}]$$

  then gives the Bismut formula

$$\boxed{(\nabla P_t f)_x v = \mathbb{E}\left[f(X_t^x)\, 1\!\!1_{\{t<\zeta(x)\}} \int_0^{t\wedge\tau(x)} \langle Q_r^* \dot{\ell}_r, dB_r \rangle \right]}$$

- This formula doesn't require any assumption on the geometry; explosion of the diffusion is allowed.